

Article

B-Coloring of Series-Parallel Graphs: Exact Cactus Values, Extremal $K_{2,r}$ Bundles, and the 2Δ Frontier

Article Info

Article history :

Received August 29, 2026
Revised October 20, 2026
Accepted October 28, 2026
Published October 30, 2026

Keywords :

B-coloring
rainbow 4-cycle
series-parallel graph
cactus graph
conflict graph
edge-coloring

Karrar Khudhair Obayes^{1*}, Ghadeer Khudhair Obayes²

¹Department of Computer Information Systems, University of Al-Qadisiyah, Al Diwaniyah, Iraq

²General Directorate of Education in Al-Qadisiyah, Al Diwaniyah, Iraq

Abstract. B-coloring strengthens proper edge-coloring by requiring every 4-cycle to be rainbow, so conflicts arise not only at common endpoints but also between opposite cycle edges. Existing planar bounds settle the twice-maximum-degree threshold in several cases, but not throughout the series-parallel class. We study this gap by combining structural arguments with exact verification through the B-conflict graph. For connected cactus graphs, we determine the B-chromatic index exactly and obtain a linear-time optimal coloring procedure. For $K_{2,r}$, we prove $qB(K_{2,r})=2r=2\Delta$, showing that the factor two cannot be reduced in a universal degree bound. Minimal-counterexample reductions then confine any unresolved obstruction to non-bipartite, low-degree series-parallel graphs in which degree-two vertices lie on interacting 4-cycles. Exact DSATUR and integer-programming models provide independently checkable finite certificates, while a separate feasibility procedure tests the 2Δ threshold on individual instances without replacing proof. The analysis isolates external 4-cycle coupling as the main structural difficulty that remains.

This is an open access article under the [CC-BY](#) license.



This is an open access article distributed under the Creative Commons 4.0 Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited. ©2026 by author.

Corresponding Author :

Karrar Khudhair Obayes

Department of Computer Information Systems, University of Al-Qadisiyah, Al Diwaniyah, Iraq

Email : karrar.khudhair@qu.edu.iq

1. Introduction

A proper edge-coloring separates colors on incident edges, and the least number of colors required is the chromatic index. B-coloring keeps this condition but adds a second one: every 4-cycle must be rainbow. The extra restriction is not captured by incidence, because opposite edges of a 4-cycle do not

share a vertex. Even so, they are not allowed to receive the same color. This small local configuration can therefore affect the size of the palette required by the whole graph.

Recent work has clarified the B-chromatic index $q_B(G)$ for several graph classes. Gyárfás, Martin, Ruszinkó, and Sárközy established $q_B(G) \leq 2\Delta + 8$ for planar graphs, $q_B(G) \leq 2\Delta$ for bipartite planar graphs, and $q_B(G) \leq \Delta + 1$ for outerplanar graphs when $\Delta \geq 4$ [1]. The planar bounds were later sharpened by Kong, Wang, and Zheng to $2\Delta + 6$ in general, $2\Delta + 4$ for $\Delta \geq 12$, and 2Δ for $\Delta \geq 38$; the same work gives $q_B(G) = \Delta$ for outerplanar graphs when $\Delta \geq 7$ [2]. Exact values are also available for many Cartesian products of paths and cycles [3], while maximal planar graphs of diameter two were considered in [4]. The results on properly colored and rainbow C_4 structures in [5] are relevant for the role played by short cycles. None of these results, however, gives a sharp description for series-parallel graphs at every degree.

The series-parallel class is a natural setting in which to study this gap. Throughout the paper we use the equivalent simple K_4 -minor-free formulation. Recent K_4 -minor-free studies use DP-coloring, strong edge-coloring, and 2-degenerate techniques [6-7]. A further related treatment is given in [8]. Constructive and structural work on K_4 -minor-free graphs provides complementary reduction tools [9-10]. A further example is provided in [11]. Related planar and sparse edge-coloring studies consider DP, list, and acyclic constraints [12-13]. Additional work in this direction appears in [14]. Richer-neighbor and Kempe-type restrictions are considered in [15-16], with a further related treatment in [17]. List-star, strong, and injective settings are represented in [18-19], with additional work in [20]. Further strong or injective variants are discussed in [21-22], with another related contribution in [23]. These results are not B-coloring bounds, but their local-reduction mechanisms are useful points of comparison because the present difficulty also appears during an extension step.

After the recent planar results, the unresolved range can be stated more precisely. The theorem in [2] already gives the 2Δ bound for every series-parallel graph with $\Delta \geq 38$, while [1] covers every bipartite member of the class. The remaining question is therefore the non-bipartite series-parallel case with $\Delta \leq 37$. A second issue is structural: one would like to explain how the class moves from the Δ -scale behavior of outerplanar graphs to the extremal 2Δ behavior of $K_{2,r}$. The obstruction appears when an edge is restored during a reduction. Its available colors are restricted not only by incident edges, but also by edges that become opposite to it on newly completed 4-cycles.

We use the B-conflict graph $B(G)$ to keep these two kinds of conflict in one object. A vertex of $B(G)$ represents an edge of G , and two such vertices are adjacent when the corresponding edges are incident or are opposite on a 4-cycle. Proposition 2.3 then gives $q_B(G) = \chi(B(G))$. This viewpoint supports both parts of the paper. Cliques in $B(G)$ give lower bounds, while the new adjacencies created during a reduction show exactly which colors become unavailable. Exact colorings of $B(G)$ also provide certificates that can be checked independently. With this framework, we derive the exact value for connected cactus graphs, obtain a linear-time construction for them, prove $q_B(K_{2,r}) = 2r = 2\Delta$, and narrow the structure of any counterexample in the remaining series-parallel case. Exact DSATUR and integer-programming formulations are used only for finite verification.

2. Method

All graphs are finite, undirected, and simple unless explicitly stated otherwise. For a graph G , we write $V(G)$ and $E(G)$ for its vertex and edge sets, $d_G(v)$ for the degree of a vertex v , and $\Delta(G)$ for the maximum degree. We use P_n , C_n , and $K_{a,b}$ for the path, cycle, and complete bipartite graph of the indicated orders. A cycle of length four is called a 4-cycle or C_4 .

2.1. Definition

A B-coloring of G is a mapping $\varphi: E(G) \rightarrow \{1, \dots, k\}$ such that (i) $\varphi(e) \neq \varphi(f)$ whenever e and f are adjacent, and (ii) the four edges of every 4-cycle receive four pairwise distinct colors. The minimum such k is denoted $q_B(G)$. The first inequality is immediate:

$$\chi'(G) \leq q_B(G) \leq |E(G)|. \quad (2.1)$$

When G has no 4-cycle, the second requirement in a B-coloring never becomes active. In that case B-coloring reduces to ordinary proper edge-coloring, and therefore $q_B(G) = \chi'(G)$.

2.2. Lemma : If G is C_4 -free, then $q_B(G) = \chi'(G)$

Proof. Every B-coloring is a proper edge-coloring, giving $\chi'(G) \leq q_B(G)$. Conversely, when G has no 4-cycle, every proper edge-coloring automatically satisfies the B-condition.

The B-conflict graph. The B-coloring condition can be expressed as an ordinary vertex-coloring problem. Let $B(G)$ be the graph whose vertices are the edges of G . Two vertices $e, f \in E(G)$ are adjacent in $B(G)$ whenever the corresponding edges share an endpoint in G or occur as opposite edges of a 4-cycle. Thus $B(G)$ records exactly the edge pairs that cannot share a color in a B-coloring.

2.3. Proposition : For every graph G , $q_B(G) = \chi(B(G))$

Proof. A vertex-coloring of $B(G)$ assigns colors to edges of G . Adjacency inherited from incidence enforces proper edge-coloring, while adjacency generated by opposite edges of a C_4 enforces four distinct colors on that cycle. The two constraints are therefore equivalent.

This representation also makes the extra difficulty of B-coloring visible. The incidence part of $B(G)$ is the usual line-graph conflict, whereas each 4-cycle adds adjacencies between opposite edges. Those additional adjacencies are sparse in some graph families but become maximally dense for $K_{2,r}$, where they force the conflict graph to be complete.

Series-parallel and cactus graphs. We work with simple series-parallel graphs in the K_4 -minor-free sense. The same convention appears in recent studies of K_4 -minor-free edge coloring, algorithmic strong coloring, weak degeneracy, and packing edge-coloring [6],[9]. Every outerplanar graph belongs to this class, but $K_{2,3}$ shows that the converse is false. We also use the standard fact that every non-null simple K_4 -minor-free graph contains a vertex of degree at most two, a feature that underlies many recent reduction arguments for the class [9-10].

A cactus graph is one in which two cycles share at most one vertex; equivalently, no edge is contained in two distinct cycles. Hence each block is either a bridge or a cycle. The associated block-cut tree allows leaf blocks to be removed one at a time, which is the structural reason an exact B-coloring formula can be obtained for this family.

A theta graph $\Theta(l_1, \dots, l_r)$ will mean two distinct nonadjacent poles joined by $r \geq 3$ internally vertex-disjoint paths, each of length at least two. With this convention, a pair of branches forms a 4-cycle precisely when both branch lengths are two.

Methodologically, the study combines the B-conflict graph transformation with structural reduction, block decomposition, local extension arguments, and exact finite verification. The theoretical part uses induction and minimal-counterexample reasoning; the computational part constructs $B(G)$, determines $\chi(B(G))$ exactly, transfers the coloring back to $E(G)$, and independently checks incidence and 4-cycle constraints.

3. Results and Discussion

3.1. Basic Bounds and Local Extension Principles

Lemma 3.1. Let H be a subgraph of G . Then $q_B(H) \leq q_B(G)$.

Proof. Restrict any B-coloring of G to $E(H)$. Every 4-cycle of H is also a 4-cycle of G , so the restriction remains a B-coloring.

Lemma 3.2. If G contains a copy of $K_{2,r}$, then $q_B(G) \geq 2r$.

Proof. Let the two vertices in the part of size two be x and y , and let $z_1, \dots, z_r$ be their common neighbors. The $2r$ edges xz_i and yz_i are pairwise B -conflicting. Edges with a common endpoint conflict by properness. If $i \neq j$, the edges xz_i and yz_j are opposite in the 4-cycle $xz_i y z_j x$. Hence they also conflict. Thus these $2r$ edges form a clique in $B(G)$.

The common-neighbor structure can yield a stronger lower bound than maximum degree alone. Set $\beta_2(G) = \max \{ |N(x) \cap N(y)| : x \neq y \}$. If x and y attain this maximum, the edges joining them to their common neighbors form a $K_{2, \beta_2(G)}$ subgraph. Lemma 3.2 then gives $qB(G) \geq 2\beta_2(G)$.

Lemma 3.3 (leaf-block extension). Let G be a cactus and let B be a leaf block with unique cut vertex x . Suppose the cactus obtained by deleting $V(B) \setminus \{x\}$ has a B -coloring from a palette P . If $|P| \geq \Delta(G)$, and additionally $|P| \geq 4$ when $B \cong C_4$, then the coloring extends over B , except that an odd cycle block with $|P| = 2$ is impossible.

Proof. Only edges of B incident with x interact by incidence with previously colored edges. No edge of B can be opposite in a 4-cycle using edges outside B , because every cycle of a cactus is contained in a single block. Thus the extension problem is local to B . If B is a bridge, at most $d_G(x) - 1 \leq |P| - 1$ colors are already forbidden at x , so one color remains. Suppose B is a cycle. The block contributes exactly two new edges incident with x , so the previously colored graph uses at most $d_G(x) - 2$ colors on edges incident with x . Since $|P| \geq \Delta(G) \geq d_G(x)$, at least two colors of P remain available for the two new edges at x . Fix two distinct such colors. The remaining edges of B form a path and can be colored successively. If $B \cong C_4$, all four cycle edges must receive distinct colors, which is possible because $|P| \geq 4$. For every other even cycle, two colors suffice when no additional restriction is present, while for an odd cycle a third color is necessary and sufficient. Hence the only obstruction under the stated hypotheses is an odd cycle with $|P| = 2$.

The extension works so cleanly because a cactus does not allow a 4-cycle to pass through two different blocks. Adding a leaf block can therefore create only local conflicts. This separation is lost in a general series-parallel graph, where 4-cycles may overlap through common poles or edges. The later difficulty with series-parallel extensions begins exactly at this point.

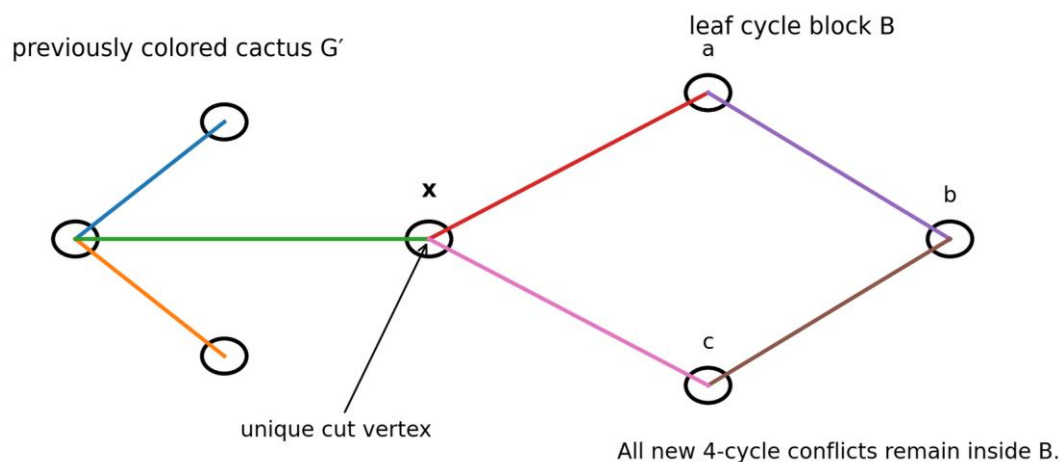


Figure 1. Leaf-block extension in a cactus. The new cycle meets the colored part only at the cut vertex x , so no 4-cycle conflict passes into an earlier block

3.2. Exact B-Coloring of Cactus Graphs

We now focus on connected cactus graphs. For a property P , write $1\{P\}=1$ when P holds and 0 otherwise, and define $M(G)=\max\{\Delta(G), 4 \cdot 1\{G \text{ contains } C_4\}, 3 \cdot 1\{\Delta(G)=2 \text{ and } G \text{ contains an odd cycle}\}\}$.

Theorem 4.1. For every connected cactus graph G with at least one edge, $q_B(G)=M(G)$.

Proof. The lower bound $q_B(G) \geq \Delta(G)$ follows from properness. If G contains C_4 , its four edges must have four distinct colors, so $q_B(G) \geq 4$. If $\Delta(G)=2$ and G contains an odd cycle, then $\chi'(G)=3$, hence $q_B(G) \geq 3$. Therefore $q_B(G) \geq M(G)$.

For the upper bound, set $k = M(G)$ and use induction on the number of blocks. If G has one block, then G is a bridge or a cycle. A bridge is trivial. For C_4 , four colors are necessary and sufficient. For any other even cycle, two colors suffice; for an odd cycle, three suffice. Thus $q_B(G) \leq k$ in every one-block case.

Assume that G has at least two blocks and choose a leaf block B with cut vertex x . Let G' be the cactus obtained by deleting $V(B) \setminus \{x\}$. The graph G' is again a connected cactus and satisfies $M(G') \leq M(G) = k$. By induction, G' has a B-coloring using colors from a fixed k -element palette P (unused colors may be added to the palette). Lemma 3.3 extends this coloring over B . Indeed $k \geq \Delta(G)$, $k \geq 4$ whenever B is a 4-cycle, and if B is odd then $k \geq 3$: when $\Delta(G)=2$ this is built into $M(G)$, while if $\Delta(G) \geq 3$ it follows from $k \geq \Delta(G)$. Hence G has a B-coloring with k colors. The theorem immediately gives several special cases that will also serve as reference points when we move to less restricted series-parallel graphs.

For cactus graphs, every B-conflict arising from a 4-cycle remains inside a single block. Apart from the C_4 case and the odd-cycle exception when $\Delta=2$, the maximum degree therefore determines the optimal palette. The edge-coloring results in [12,19] concern different conflict models, so their numerical bounds are not used here. Their relevance is methodological: decomposition arguments are particularly effective when the additional coloring restrictions remain local. Parallel composition in a general series-parallel graph is precisely what can destroy that locality.

Corollary 4.2. If G is a cactus with $\Delta(G) \geq 4$, then $q_B(G) = \Delta(G)$.

Corollary 4.3. If G is a C_4 -free cactus, then $q_B(G) = \chi'(G)$; in particular, $q_B(G) = \Delta(G)$ whenever $\Delta(G) \geq 3$.

Corollary 4.4. For a cycle C_n , $q_B(C_n)=4$ if $n=4$, $q_B(C_n)=3$ if n is odd, and $q_B(C_n)=2$ if n is even and $n \neq 4$.

The cycle formulas also clarify why C_4 has to be separated from the other even cycles. Ordinary edge-coloring treats even cycles in the same way, but B-coloring introduces an extra restriction only when the cycle has length four: opposite edges must receive different colors. In a cactus that restriction is confined to one block, whereas in a more general series-parallel construction it may interact with a neighboring parallel structure.

3.2.1. A Constructive Algorithm

The proof of Theorem 4.1 is constructive. Build the block-cut tree of G in $O(|V| + |E|)$ time and choose any block as the root. Color that block optimally, then process the remaining blocks according to the rooted tree, retaining at each cut vertex the colors already used on incident edges. Since every block is processed once, the construction is linear.

Proposition 4.5. There is an $O(|V(G)| + |E(G)|)$ -time algorithm that computes and constructs an optimal B-coloring for every cactus graph G .

Proof. The block-cut tree and all blocks of a cactus are obtained by depth-first search in $O(|V(G)| + |E(G)|)$ time. The algorithm described above processes each block and each edge exactly once, and the local extension at a cut vertex uses constant-time color-marking operations per incidence. The resulting coloring uses exactly $M(G)$ colors by Theorem 4.1, and is therefore optimal.

3.3. Extremal Parallel-Path Bundles and the Factor Two

Among theta graphs, the strongest concentration of B-conflicts occurs when all branches have length two. This gives $K_{2,r}$, with two poles joined through r common degree-two vertices. The graph is series-parallel, and it provides the basic extremal family behind the factor two in the degree bound.

Theorem 5.1. For every integer $r \geq 2$, $q_B(K_{2,r}) = 2r = 2\Delta(K_{2,r})$.

Proof. The graph $K_{2,r}$ has exactly $2r$ edges. By Lemma 3.2 all of them are pairwise adjacent in the B-conflict graph, so $q_B(K_{2,r}) \geq 2r$. Assigning a distinct color to every edge is a B-coloring, giving equality. Since the two vertices in the part of size two have degree r , $\Delta(K_{2,r}) = r$. Figure 2 shows the smallest non-outerplanar member of this family, $K_{2,3}$, together with its B-conflict graph.

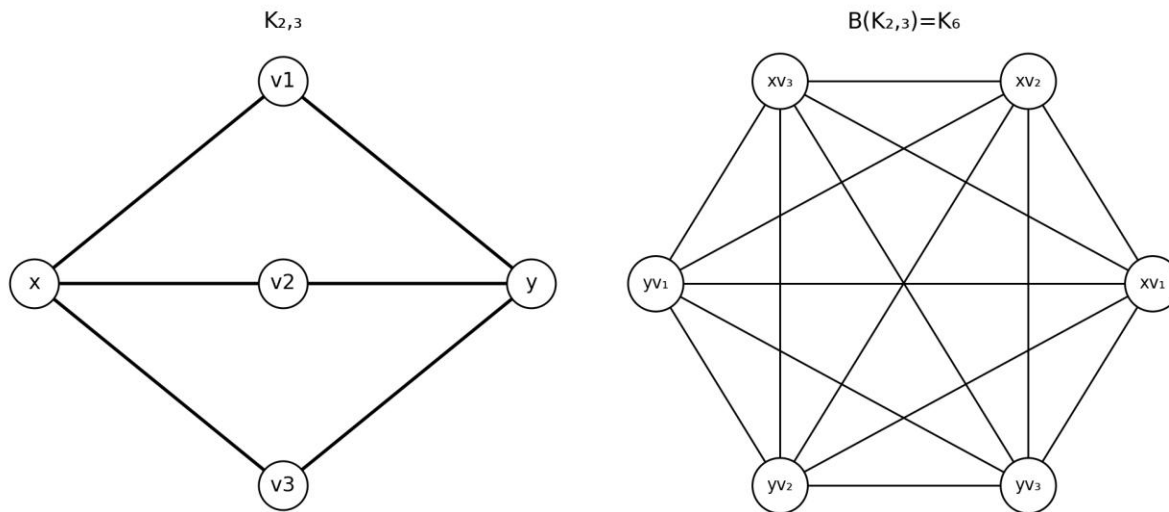


Figure 2. The extremal bundle $K_{2,3}$ and its B-conflict graph. Every pair of the six branch edges is B-conflicting, so the conflict graph is complete and $q_B(K_{2,3})=6$.

Corollary 5.2. If a constant c satisfies $q_B(G) \leq c\Delta(G) + O(1)$ for every series-parallel graph G , then $c \geq 2$. The factor two is therefore unavoidable before one even attempts a general upper bound. This is a genuine change from the outerplanar setting, where $q_B(G) \leq \Delta + 1$ for $\Delta \geq 4$ [1] and $q_B(G) = \Delta$ for $\Delta \geq 7$ [2].

The equality $q_B(K_{2,r}) = 2\Delta$ pinpoints where this factor comes from. Any two of the $2r$ edges are incompatible: they either meet at a pole or are opposite on a 4-cycle. Thus $B(K_{2,r})$ is a complete graph, so the lower bound is attained. Dense local conflict patterns also appear in strong and star edge-coloring of sparse or K_4 -minor-free graphs [24-25], although the underlying coloring rules are different. For B-coloring the implication is specific: a proof of a universal 2Δ bound must accommodate bundles that can use the entire palette.

3.3.1. Subdivided Parallel Paths

Subdividing the length-two branches weakens this extremal behavior. B-conflicts between nonincident edges are created only by actual 4-cycles, not by longer even cycles. In $\Theta(l_1, \dots, l_r)$, branches i and j form a cycle of length $l_i + l_j$. Since each $l_i \geq 2$, a 4-cycle occurs exactly when $l_i = l_j = 2$. Thus a theta graph may contain many parallel pole-to-pole routes without producing many opposite-edge conflicts.

Proposition 5.3. Let $\Theta = \Theta(l_1, \dots, l_r)$ be a theta graph in the sense of Section 2. Then Θ contains a 4-cycle if and only if two branch lengths are equal to two. In particular, if at most one branch has length two, then Θ is C_4 -free and $q_B(\Theta) = \chi'(\Theta)$.

Proof. The union of two internally vertex-disjoint pole-to-pole branches of lengths l_i and l_j is a cycle of length $l_i + l_j$. Because every branch length is at least two, this cycle has length four exactly when $l_i = l_j = 2$. The final statement follows from Lemma 2.2.

3.4. Toward the Sharp 2Δ Bound for Series-Parallel Graphs

These exact cases suggest the following bound for the full series-parallel class.

Conjecture 6.1. Every simple series-parallel graph G satisfies $q_B(G) \leq 2\Delta(G)$.

The bound is sharp because of Theorem 5.1, but large parts of the class are already settled. Every bipartite series-parallel graph satisfies $q_B(G) \leq 2\Delta$ by the bipartite planar theorem [1]. In addition, the planar result of [2] gives the same bound whenever $\Delta \geq 38$. Hence a counterexample to Conjecture 6.1, if one exists, must be non-bipartite and have $\Delta \leq 37$. This is exactly the range in which odd cycles or triangles may occur together with the parallel structures that generate 4-cycles.

Reduction arguments for K_4 -minor-free graphs [6],[9] provide a natural framework for this remaining range, but the extension step is different for B-coloring. When an edge e is restored in an ordinary edge-coloring, one avoids the colors on edges incident with e . Here one must also avoid colors on edges that become opposite to e in a newly created 4-cycle. A low-degree reduction may therefore identify a useful configuration without, by itself, controlling all colors forbidden by C_4 interactions.

3.4.1. A Safe Extension Criterion

Let $e=uv$ be uncolored in a partial B-coloring. Write $F(e)$ for the set of colors already appearing on edges that either meet e or would be opposite to e on a 4-cycle after e is inserted. Any color outside $F(e)$ is safe for e , because neither type of B-conflict is created.

Lemma 6.2. If a partial B-coloring uses a palette P and $|F(e)| < |P|$, then the coloring extends to e .

Proof. Choose any color in P that is not in $F(e)$. By construction it conflicts neither by incidence nor by a 4-cycle opposition.

If Conjecture 6.1 is false, the results in [1,2] show that a counterexample must be non-bipartite and satisfy $\Delta \leq 37$. Fix one with minimum $|V| + |E|$ and write $\Delta = \Delta(G)$. We next record the restrictions forced by this minimal choice.

3.4.2. Minimal-Counterexample Restrictions

Let G be a counterexample minimizing $|V(G)| + |E(G)|$. The known planar bounds imply that G is non-bipartite and $\Delta(G) \leq 37$ [1-2]. The purpose of the next reductions is to identify what such a smallest graph would have to look like in the only range that remains open.

Proposition 6.3. A minimal counterexample G is connected and has minimum degree at least two.

Proof. Connectivity follows by coloring components independently with a common palette of 2Δ colors. If v has degree one with incident edge e , color $G-e$ by minimality. On reinserting e , no 4-cycle contains e , and at most $\Delta-1$ colors occur on edges adjacent to e at its other endpoint. Since $2\Delta > \Delta-1$, a safe color exists.

Proposition 6.4. A minimal counterexample contains a vertex of degree two.

Proof. Every non-null simple K_4 -minor-free graph has a vertex of degree at most two, a structural property used in recent coloring analyses and algorithms for this class [9-10]. Proposition 6.3 excludes degrees zero and one, so a minimal counterexample must contain a vertex of degree two.

Take a 2-vertex v with neighbors x and y . If xy is not an edge, then after deleting v the new B-conflicts created when xv and vy are restored can only come from common neighbors of x and y . If xy is present, the triangle xvy may interact with other length-two x - y paths. It is therefore useful to distinguish degree-two vertices by whether they lie in a triangle and by the size of $N(x) \cap N(y)$.

Lemma 6.5. Let v be a 2-vertex with neighbors x, y . Suppose v is not contained in a 4-cycle of G . Then every B -coloring of $G-v$ with 2Δ colors extends over xv and vy .

Proof. Because v lies in no 4-cycle, neither xv nor vy acquires an opposite-edge conflict. To color xv , avoid colors on edges incident with x ; at most $\Delta-1$ colors are forbidden. After coloring xv , color vy while avoiding colors incident with y and the color on xv ; at most Δ colors are forbidden. Since the palette has 2Δ colors, both choices are possible.

Corollary 6.6. In a minimal counterexample, every degree-two vertex lies on a 4-cycle.

Corollary 6.6 sharply narrows the possible local structure. A 2-vertex v with neighbors x and y lies on a 4-cycle exactly when x and y have another common neighbor. Thus every degree-two vertex in a minimal counterexample must belong to a bundle containing at least two length-two x - y paths. These are the same local configurations that produce the extremal family $K_{2,r}$.

3.4.3. Why the Remaining Configuration Is Delicate

Let x and y have t common degree-two neighbors $v_1, \dots, v_t$. The corresponding $2t$ branch edges form a clique in $B(G)$, so those edges must all receive different colors. Because $t \leq \min\{d(x), d(y)\} \leq \Delta$, this demand by itself does not exceed the 2Δ palette. The difficulty comes from edges outside the bundle. An external edge at x may forbid a color on the x -side but not on the y -side, unless another 4-cycle couples it across the bundle. A proof should use this asymmetry rather than reserve two disjoint Δ -sized palettes, which would leave no flexibility at all.

Lemma 6.7. Let $S = \{v_1, \dots, v_t\}$ be degree-two vertices with $N(v_i) = \{x, y\}$ for every i , and let $H = G - S$. Fix a palette P of 2Δ colors, where $\Delta = \Delta(G)$. Every B -coloring of H from P extends to all $2t$ edges xv_i, yv_i provided no edge outside the bundle becomes opposite, in a 4-cycle, to a bundle edge. In particular, the extension always exists when every 4-cycle containing a bundle edge is generated by two vertices of S .

Proof. Let A be the colors used on edges of H incident with x and let B be the colors used on edges of H incident with y . Since t bundle edges are missing at each pole, $|A| \leq \Delta - t$ and $|B| \leq \Delta - t$. The t edges xv_i may use colors in $P \setminus A$ and the t edges yv_i may use colors in $P \setminus B$, and all $2t$ bundle edges must receive distinct colors because they form a clique in the B -conflict graph. Apply Hall's theorem to these $2t$ color lists. A subfamily lying entirely on one side has a union of size at least $2\Delta - (\Delta - t) = \Delta + t \geq 2t$. A mixed subfamily has union $P \setminus (A \cap B)$, whose size is also at least $\Delta + t \geq 2t$. Therefore a system of distinct representatives exists. The stated hypothesis excludes any further opposite-edge restrictions, so the resulting assignment extends the B -coloring of H .

Example 6.8 (external coupling). Let $v_1, \dots, v_t$ be degree-two common neighbors of x and y , and let z be another common neighbor with $d(z) \geq 3$. For each i , the cycle $x-v_i-y-z$ makes xv_i opposite to yz and yv_i opposite to xz . The colors on xz and yz are therefore excluded from entire sides of the degree-two bundle. Several external common neighbors, or overlapping 4-cycles in the surrounding graph, can produce correlated losses from the lists used in Lemma 6.7. This is why the Hall argument for an isolated $K_{2,t}$ bundle does not automatically extend to an arbitrary series-parallel graph.

This example points to two possible ways of handling the remaining interaction. One is to recolor along an alternating conflict chain in $B(G)$, using a two-color interchange to free a needed color at one pole before the Hall assignment. Similar recoloring ideas appear in recent list and injective coloring work [22-23], although $B(G)$ has the additional adjacencies created by opposite edges of 4-cycles. A second route is structural. One could seek, by discharging or a reducible-configuration argument, a degree-two bundle whose interaction with the rest of the graph is sufficiently restricted. Comparable

strategies have been useful in recent sparse and K_4 -minor-free coloring proofs [6],[19], although here the extra conflicts between opposite edges of 4-cycles must still be tracked explicitly.

Remark 6.9. Lemma 6.7 shows that the isolated bundle is not itself the source of failure: a pure $K_{2,t}$ bundle can be colored within the available 2Δ colors. The unresolved part is the collection of 4-cycles that links bundle edges to colors already used outside the bundle. Thus $K_{2,t}$ explains why the factor two is sharp, while Example 6.8 identifies the additional coupling that prevents the same counting argument from settling Conjecture 6.1.

These two routes may be more useful in combination. A structural reduction could first locate a bundle whose external interactions are limited; a short alternating chain in $B(G)$ could then be used to make a needed color available. In this division of roles, the structural step reduces the number of outside restrictions and the recoloring step deals with the remaining color conflict. Comparable ideas occur in recent sparse-graph coloring work [6],[17], but here they remain a possible route rather than a completed proof. This leaves a concrete question: must every minimal counterexample contain a coupled bundle that can be repaired either by a bounded alternating-chain recoloring or by deleting a reducible configuration and extending the coloring back? A uniform proof of either alternative would complete the reduction program for the proposed 2Δ bound.

3.5. Computational Verification and Reproducibility

The computational part of the study is designed to mirror the structural analysis rather than run separately from it. We first specify the graph class, notation, target bounds, and structural parameters. Finite instances come either from series-parallel decompositions or from the explicit families studied above. For each graph we record $|V|$, $|E|$, Δ , $\beta_2(G)$, block information, $\omega(B(G))$, $\chi(B(G))=qB(G)$, and the density of B-conflicts. These entries are fixed graph invariants, not sampled data.

The theoretical analysis uses the extension lemmas, cactus block structure, extremal bundles, and minimal-counterexample reductions developed in the previous sections. For the computational check, $B(G)$ is built explicitly and colored exactly with DSATUR branch-and-bound or the integer-programming model; the resulting colors are then transferred back to $E(G)$. The implementation therefore needs a graph representation, block-cut data, a routine for constructing $B(G)$, an exact coloring solver, and an independent certificate check. Using $B(G)$ on both sides makes the connection transparent: every color restriction in an extension argument appears as an edge of the conflict graph.

Each output is judged against a proved formula, a lower bound, or the proposed 2Δ threshold. A candidate coloring is verified again on G , checking both incidence and all 4-cycles, and clique certificates in $B(G)$ are retained as lower bounds. For a general series-parallel input, the key finite question is whether a coloring with at most 2Δ colors exists. The unresolved cases of greatest interest are non-bipartite graphs with $\Delta \leq 37$, since [1-2] cover the remaining regimes. A successful run returns a checked coloring; a failed feasibility test is followed by exact coloring to determine what happens on that specific graph. The computation is therefore used for certification and diagnosis, not as a replacement for proof.

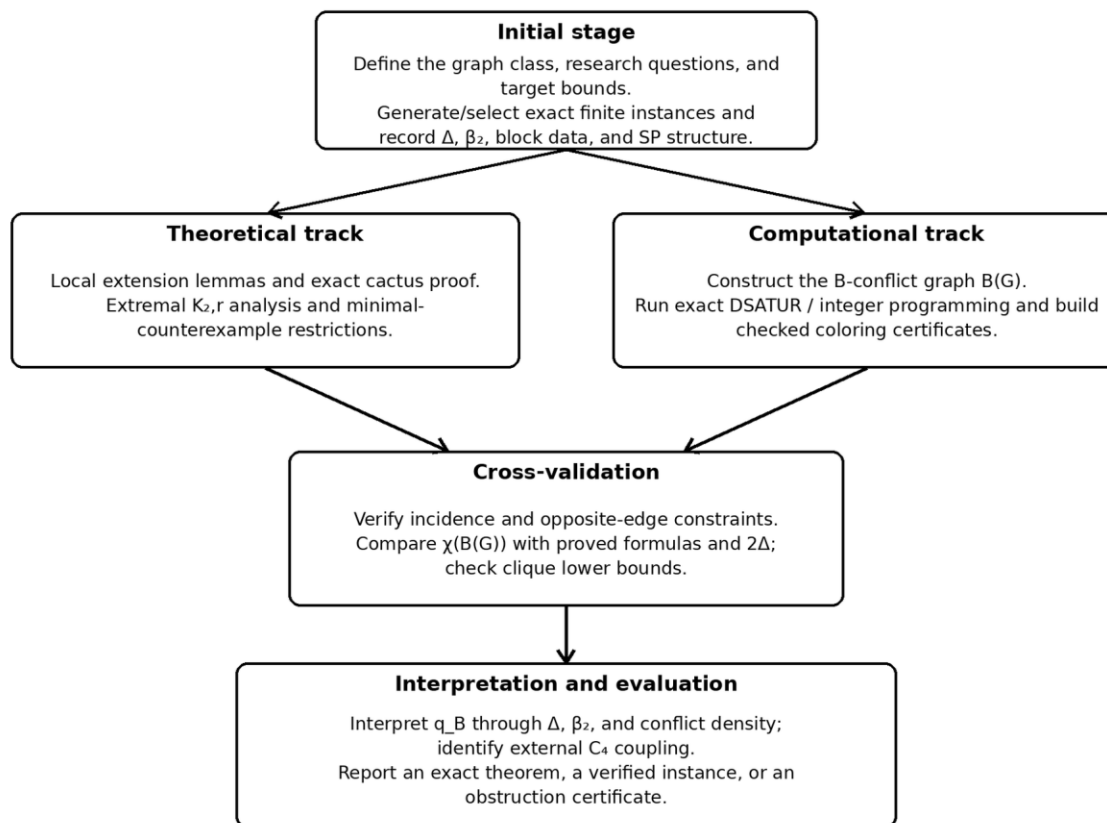


Figure 3. Workflow connecting the structural analysis with exact coloring, certificate verification, and final interpretation.

Table 1. Research stages, computational tools, and criteria used for verification

Stage	Input / data	Technique and instrument	Evaluation indicator
Initial	Series-parallel or benchmark instances; Δ , β_2 , block information	Generate/recognize SP graphs; adjacency lists; block-cut tree	Graph class verified; structural data recorded
Implementation: theory	Cactus blocks, bundles, and degree-two configurations	Induction, local extension, Hall's theorem, minimal-counterexample analysis	Exact formula, lower bound, or reducible configuration established
Implementation: computation	Graph G together with B-conflict relations	Construct $B(G)$; exact DSATUR; integer programming	$\chi(B(G))$, clique lower bound, explicit coloring certificate
Final evaluation	Theoretical results and computational certificates	Direct certificate check; comparison with formulas and 2Δ	Exact agreement, 2Δ -feasibility, or a certified obstruction

Proposition 2.3 leads to a straightforward exact check on any finite instance. Given a series-parallel graph G , we construct $B(G)$, determine $\chi(B(G))$, transfer that coloring to the corresponding edges of G , and then verify the B-coloring directly on the original graph. Treating construction, optimization, and certificate verification as separate steps makes the computation easier to audit independently.

Algorithm 1. Exact B-Coloring Verification through the B-Conflict Graph

- 0: Input: finite simple graph $G = (V, E)$.
- 1: Phase I: Construct the B-conflict graph.
- 2: Set $V(B(G)) \leftarrow E(G)$, with one conflict vertex for each edge of G .
- 3: Add a conflict for every pair of incident edges.
- 4: Enumerate the 4-cycles of G .
- 5: For each 4-cycle, add the two conflicts between opposite edge pairs.
- 6: Phase II: Compute an exact coloring.
- 7: Color $B(G)$ exactly by DSATUR branch-and-bound.
- 8: Set $q_B(G) \leftarrow \chi(B(G))$.
- 9: Phase III: Verify the certificate.
- 10: Transfer each conflict-graph color to its corresponding edge of G .
- 11: Confirm that incident edges have different colors and inspect each 4-cycle for four distinct colors.
- 12: return the exact value $q_B(G)$ together with its checked B-coloring certificate.

Once the exact-coloring routine and its tie-breaking convention are fixed, the procedure is deterministic. Besides $q_B(G)$, it returns an explicit edge-coloring, so the reported value can be checked directly on G .

Construction of $B(G)$ begins with the ordinary incidence conflicts. For each 4-cycle $e_1e_2e_3e_4$ in cyclic order, we then insert the two opposite pairs e_1e_3 and e_2e_4 , omitting duplicates. Direct cycle enumeration is adequate for small graphs. For larger sparse instances, one can instead intersect neighborhoods: if x and y have two common neighbors u and v , then $x-u-y-v-x$ is a 4-cycle and supplies the corresponding opposite-edge conflicts.

Algorithm 2. Constructive Optimal B-Coloring of a Connected Cactus Graph

- 0: Input: a connected cactus $G = (V, E)$.
- 1: Phase I: Obtain the cactus block decomposition.
- 2: Construct the block-cut tree T and choose a root block.
- 3: Determine the target palette size from the exact cactus formula.
- 4: Phase II: Extend the coloring.
- 5: Color the root block with the target palette.
- 6: for each remaining block X in leaf-to-root order do
- 7: Keep the colors already fixed at the shared cut vertex.
- 8: If X is a bridge, assign an available color distinct from incident colors.

Algorithm 2. Constructive Optimal B-Coloring of a Connected Cactus Graph

- 9: If X is a cycle, extend the coloring around X while preserving properness.
- 10: If $X = C_4$, assign four pairwise distinct edge colors.
- 11: If $\Delta = 2$ and X is odd, use the required three-color cycle pattern.
- 12: end for
- 13: Phase III: Validate the construction.
- 14: Check all incidence conflicts and opposite-edge conflicts on 4-cycles.
- 15: return the optimal B-coloring and its palette size.

Proposition 7.1. Let G be a simple K_4 -minor-free graph with $n = |V(G)|$ and maximum degree Δ . Then $B(G)$ can be constructed in $O(n\Delta^2)$ time and $O(n\Delta^2)$ space using adjacency lists, constant-time adjacency lookup, and duplicate-safe conflict insertion.

Proof. A simple K_4 -minor-free graph is 2-degenerate and has at most $2n-3$ edges for $n \geq 2$. Incidence conflicts can be generated by processing, at each vertex v , all pairs of incident edges; the total number of such checks is at most $O(n\Delta^2)$. For opposite-edge conflicts, process each edge uv and all choices $x \in N(u) \setminus \{v\}$ and $y \in N(v) \setminus \{u\}$; whenever $xy \in E(G)$, the edges uv and xy are opposite in the 4-cycle $u-x-y-v-u$. There are at most $(d(u)-1)(d(v)-1) \leq \Delta^2$ choices per edge. Since $|E(G)| = O(n)$, the total work is $O(n\Delta^2)$, and duplicate-safe insertion gives the same asymptotic space bound.

3.5.1. Exact Coloring and Certificates

Once $B(G)$ is available, its chromatic number is computed exactly by DSATUR branch-and-bound or by the integer-programming model below. The resulting vertex-coloring of $B(G)$ is immediately a B-coloring certificate for G . Independent clique searches are retained as lower-bound certificates; in $K_{2,r}$ this includes the clique on all $2r$ branch edges. Keeping graph construction, optimization, and certificate checking as separate tasks also helps isolate implementation errors.

As a second exact formulation, choose a candidate palette $C = \{1, \dots, U\}$. For $e \in E(G)$ and $c \in C$, let $x(e, c) = 1$ when edge e is assigned color c , and let $y(c) = 1$ when color c is used. Since the edges of $B(G)$ are exactly the forbidden equal-color pairs, $qB(G) = \chi(B(G))$ is obtained from the following model:

$$\begin{array}{ll}
 \text{minimize} & \sum y(c) \text{ over } c \in C \\
 \text{subject to} & \sum x(e, c) \text{ over } c \in C = 1 \quad \text{for every } e \in E(G), \\
 & x(e, c) + x(f, c) \leq 1 \quad \text{for every } \{e, f\} \in E(B(G)) \text{ and } c \in C, \\
 & x(e, c) \leq y(c) \quad \text{for every } e \in E(G) \text{ and } c \in C, \\
 & y(c) \geq y(c+1) \quad \text{for } c = 1, \dots, U-1, \\
 & x(e, c), y(c) \in \{0, 1\}.
 \end{array}$$

The first family of constraints gives each edge one color. The second prevents any conflicting pair in $B(G)$ from sharing a color, the third links assigned colors to the activation variables, and the fourth only removes color-label symmetry. Choosing $U = |E(G)|$ yields an exact optimization model. For a finite test of Conjecture 6.1, one instead sets $U = 2\Delta$ and solves the feasibility problem; any feasible solution is already a verifiable 2Δ B-coloring.

Algorithm 3. Constructive 2Δ Feasibility and Counterexample Certifier

```

0:   Input: a finite simple series-parallel graph  $G$ .
1:   Compute  $\Delta \leftarrow \Delta(G)$  and set  $U \leftarrow 2\Delta$ .
2:   Construct the B-conflict graph  $B(G)$ .
3:   Solve the  $U$ -color feasibility problem for  $B(G)$  by exact DSATUR or the IP constraints above.
4:   if a feasible coloring is found then
5:     Transfer the colors to  $E(G)$  and verify all incidence and  $C_4$  constraints.
6:     return a checked B-coloring of  $G$  using at most  $2\Delta$  colors.
7:   else
8:     Continue the exact coloring search to determine  $\chi(B(G))$ .
9:     Produce a clique/lower-bound record and an optimal coloring certificate.
10:  return  $q_B(G) = \chi(B(G)) > 2\Delta$ ; this finite graph would be a counterexample to Conjecture 6.1.
11:  end if

```

Algorithm 3 gives a definite answer for each finite graph tested. If a 2Δ coloring exists, it returns and verifies one. If the feasibility test fails, the exact search is continued until $\chi(B(G))$ is known, which would certify a finite counterexample if $\chi(B(G)) > 2\Delta$. No assumption about the conjecture is built into the procedure. The search can of course be exponential in the worst case; here its role is exact certification rather than a polynomial-time solution of the unresolved class.

3.5.2. Algorithmic Interpretation

The algorithms have distinct roles. Algorithm 1 is used as an exact checker on bounded instances. Algorithm 2 implements the constructive proof of Theorem 4.1 and therefore avoids general optimization on cactus graphs. Algorithm 3 first asks the question relevant to the open case whether 2Δ colors suffice and invokes full exact coloring only if that test fails. Keeping these tasks separate distinguishes a proved construction, a general verifier, and a conjecture-oriented search.

3.5.3. Reproducibility Protocol

For reproducibility, each generated graph is stored in a canonical representation together with its order, size, Δ , $\beta_2(G)$, the computed $q_B(G)$, a largest-clique certificate in $B(G)$, and an explicit optimal coloring. Isomorphic duplicates are removed before summaries are formed. A complete record also specifies how the graph was generated or decomposed, the adopted K_4 -minor-free convention, the isomorphism procedure, the DSATUR tie-breaking rule or IP settings, and the final certificate checker. These details are sufficient for another implementation to reconstruct the instance and verify the claimed coloring value.

3.5.4. Evaluation Baseline and Comparative Bounds

Direct B-coloring results provide the main benchmark for comparison. Gyárfás et al. obtained $q_B(G) \leq 2\Delta + 8$ for planar graphs, $q_B(G) \leq 2\Delta$ for bipartite planar graphs, and $q_B(G) \leq \Delta + 1$ for outerplanar graphs with $\Delta \geq 4$ [1]. Kong, Wang, and Zheng improved the planar bounds to $2\Delta + 6$ in general, $2\Delta + 4$ for $\Delta \geq 12$, and 2Δ for $\Delta \geq 38$, and also proved $q_B(G) = \Delta$ for outerplanar graphs when $\Delta \geq 7$ [2]. Exact values are known for many Cartesian products [3], and [4] treats maximal planar graphs of diameter two. Against this background, Theorem 5.1 shows that the series-parallel family already contains graphs with $q_B = 2\Delta$, so the coefficient two cannot be reduced universally.

Table 2. Comparison with recent B-coloring bounds and the present exact results

Graph Class	Representative bound/result	Conditions	Sharpness Mechanism	Source
Outerplanar	$q_B(G)=\Delta$	outerplanar, $\Delta \geq 7$	limited C_4 interaction in outerplanar structure	[2]
Bipartite planar	$q_B(G) \leq 2\Delta$	all bipartite planar graphs	factor two compatible with planar C_4 structure	[1]
Planar	$q_B(G) \leq 2\Delta + 6$; $\leq 2\Delta + 4$ if $\Delta \geq 12$; $\leq 2\Delta$ if $\Delta \geq 38$	planar graphs	general planar upper bound	[2]
Cactus	exact formula $M(G)$	connected, at least one edge	C_4 / odd-cycle / degree exceptions	Theorem 4.1
Series-parallel	$q_B(K_{2,r}) = 2\Delta$	$K_{2,r}$, $r \geq 2$	complete conflict graph on the $2r$ branch edges	Theorem 5.1
Series-parallel	$q_B(G) \leq 2\Delta$ conjectured	all simple SP graphs	would be sharp because of $K_{2,r}$	Conjecture 6.1

The remaining edge-coloring papers are used as methodological comparisons rather than as numerical bounds for q_B . Palette-index and product-graph studies illustrate how global color patterns can arise from local edge constraints [26-27]. Distinguishing and adjacent-vertex-distinguishing variants provide related examples of additional color-separation requirements [28-29], while near-interval edge coloring gives a further structured palette comparison [30]. Since those problems forbid different pairs of edges, their numerical results are not transferred here. What carries over is the style of structural and algorithmic reasoning.

Combining [1] and [2] leaves a sharply defined unresolved range. Bipartite series-parallel graphs already satisfy the proposed bound, and every series-parallel graph with $\Delta \geq 38$ is covered by the planar result. Thus a counterexample to Conjecture 6.1 could occur only when the graph is non-bipartite and $\Delta \leq 37$. A triangle does not itself introduce an opposite-edge conflict, but it can change the local structure and the colors available near bundles of two-edge paths.

3.6. Structural Interpretation

3.6.1. Exact Cactus Regime: Locality Produces a Constructive Formula

Theorem 4.1 places cactus graphs in a genuinely local regime. Every cycle lies inside one block, so an opposite-edge conflict cannot cross a block boundary. The optimal palette is therefore governed by Δ except for C_4 and the odd-cycle case when $\Delta=2$. This locality is also why the proof becomes the $O(|V|+|E|)$ construction of Proposition 4.5. Similar sparse-graph coloring arguments are most effective when the additional constraints remain confined to the local piece being extended [8],[19].

This behavior contrasts with general series-parallel graphs. In several DP, strong, list, and injective edge-coloring settings, one can preserve a reduction by controlling conflicts inside a bounded neighborhood [13],[20]. For B-coloring, however, the new conflict may appear only after a particular 4-cycle is completed. The difficulty is therefore not simply a large degree: parallel composition can make edges from formerly separate local pieces become opposite on a 4-cycle.

3.6.2. Quantitative Transition through $\beta_2(G)$

Here $\beta_2(G)$ records the maximum number of common neighbors of a pair of vertices, equivalently the largest collection of internally disjoint two-edge paths with the same poles. Lemma 3.2 yields $q_B(G) \geq 2\beta_2(G)$, and equality occurs for $K_{2,r}$, where $\beta_2=r=\Delta$ and $q_B=2\Delta$. The parameter does not determine q_B on its own, however. The exact examples in Table 3 show that increasing β_2 can raise

the conflict demand, while graphs with the same β_2 can still have different q_B/Δ values because maximum degree, odd cycles, and external coupling contribute in different ways.

Table 3. Exact certificates comparing Δ , $\beta_2(G)$, $q_B(G)$, and the resulting q_B/Δ ratio

Representative instance	Δ	$\beta_2(G)$	$q_B(G)$	q_B/Δ	Interpretation
P_6	2	1	2	1.00	C_4 -free baseline
C_4	2	2	4	2.00	single local C_4 reaches factor two
C_5	2	1	3	1.50	odd-cycle edge-coloring effect
$\Theta(2,3,3)$	3	1	3	1.00	parallel routes without a C_4 bundle
$\Theta(2,2,3)$	3	2	4	1.33	one two-branch C_4 bundle
$K_{2,3}$	3	3	6	2.00	complete B-conflict graph
$K_{2,4}$	4	4	8	2.00	extremal bundle persists

Table 3 shows both what β_2 captures and what it misses. For the complete two-edge bundle, $\beta_2=\Delta$ and $q_B/\Delta=2$, exactly the extremal $K_{2,r}$ behavior. By contrast, C_4 and C_5 differ at $\Delta=2$ for a reason that β_2 alone does not encode, and $\Theta(2,2,3)$ has $\beta_2=2$ without requiring the full 2Δ palette. This suggests that a bound involving Δ and β_2 may be informative, but any sharp form must also describe how the relevant 4-cycles overlap among themselves and with external edges.

3.6.3. External C_4 Coupling and the 2Δ Frontier

Theorem 5.1 and Lemma 6.7 separate the internal color demand of a bundle from the extra restrictions created outside it. A pure $K_{2,t}$ bundle needs $2t$ distinct colors on its branch edges, yet those colors still fit within a 2Δ palette. The obstruction appears when further common neighbors or overlapping 4-cycles rule out colors already used elsewhere, as in Example 6.8. The bipartite theorem [1] settles this issue for all bipartite series-parallel graphs, and the planar result [2] covers every case with $\Delta \geq 38$. Any unresolved example must therefore lie in the non-bipartite, lower-degree range.

The two directions proposed in Section 3.4 can be tested on explicit instances. Alternating conflict chains address the immediate problem of color availability in $B(G)$, whereas a discharging or reducible-configuration argument would aim to produce a bundle with limited outside interaction. Both recoloring and sparse-structure reductions have been successful in related list and K_4 -minor-free coloring problems [6],[12]. The additional burden here is that every step must preserve separation of opposite edges on all 4-cycles. A complete proof for $\Delta \leq 37$ may therefore need a structural reduction together with a local recoloring argument.

3.6.4. List B-Coloring under External Coupling

In the list version, each edge e has its own admissible color set and one seeks a B-coloring using only those colors. Equivalently, this is list coloring of $B(G)$. Let $ch_B(G)$ be the least k such that every assignment of k -element lists to the edges of G admits a B-coloring. External 4-cycle coupling already removes common colors from the two sides of a bundle in the ordinary problem; arbitrary lists can make that loss asymmetric before the extension even starts. Recent list, DP, star-list, and injective edge-coloring results show the extra difficulty created by edge-dependent color sets [18],[20]. In the present setting each 4-cycle adds further non-incidence restrictions to those lists.

Proposition 8.1. For every integer $r \geq 2$, $ch_B(K_{2,r}) = 2r$.

Proof. The B-conflict graph of $K_{2,r}$ is the complete graph on $2r$ vertices. Therefore its list chromatic number equals $2r$, and the equality follows from the conflict-graph formulation.

It remains natural to ask whether an analogue of the 2Δ bound holds for list B-coloring of series-parallel graphs. Such a statement would imply Conjecture 6.1, but the symmetry used in Lemma 6.7 disappears: different bundle edges may lose different colors because of both their lists and the surrounding 4-cycles. A list version would therefore require a Hall condition for the actual reduced lists, or a recoloring theorem strong enough to repair them. Thus the list problem is not a separate obstruction; it is the same external-coupling difficulty with less symmetry in the available colors.

3.6.5. Remaining Structural Problem: Joint Dependence on Δ and β_2

Table 3 also clarifies the role of $\beta_2(G)$. It measures how densely two-edge paths are concentrated around a pair of poles and immediately gives the lower bound $q_B(G) \geq 2\beta_2(G)$; Δ , in contrast, measures ordinary incidence demand. Cactus graphs have $\beta_2 \leq 2$, whereas $K_{2,\Delta}$ has $\beta_2 = \Delta$ and reaches $q_B = 2\Delta$. Intermediate theta graphs show that identical β_2 values need not produce the same q_B/Δ ratio. Any sharper parameterization therefore has to retain some information about the overlap pattern of the corresponding 4-cycles.

Problem 8.2. Determine the smallest function $f(\Delta, \beta_2)$ such that $q_B(G) \leq f(\Delta(G), \beta_2(G))$ for every simple series-parallel graph G , and identify additional structural information needed when Δ and β_2 alone do not determine a sharp bound.

A two-parameter bound in Δ and β_2 would connect the cactus and $K_{2,\Delta}$ ends of the class more closely than a degree-only estimate. The examples above indicate, however, that another structural quantity may be needed to distinguish different patterns of 4-cycle overlap. The exact certificate procedure from Section 3.5 gives a practical way to search for graphs sharing the same pair (Δ, β_2) but having different q_B values; such pairs would show directly where the extra information becomes necessary.

4. Conclusion

Two exact regimes emerge from the analysis. For a connected cactus, $q_B(G)$ is fixed by the maximum degree together with the local C_4 and odd-cycle exceptions, and the corresponding optimal coloring is obtained in $O(|V| + |E|)$ time. At the opposite end, $q_B(K_{2,r}) = 2r = 2\Delta$, so a universal degree bound cannot have coefficient smaller than two. The parameter $\beta_2(G)$ helps explain this difference: cactus graphs satisfy $\beta_2 \leq 2$, whereas $K_{2,\Delta}$ has $\beta_2 = \Delta$ and all 2Δ branch edges form a clique in $B(G)$.

What remains open is the non-bipartite series-parallel case with maximum degree at most 37. The minimal-counterexample reductions force every degree-two vertex onto a 4-cycle, while Lemma 6.7 shows that an isolated two-edge bundle can still be extended within the target palette. The unresolved obstruction is the coupling produced by 4-cycles that connect the bundle to colors used elsewhere. The integer-programming model, exact conflict-graph search, and 2Δ feasibility procedure allow this mechanism to be checked on finite graphs without treating computation as a proof. A full resolution now requires control of coupled bundles, most likely through a reducible configuration together with an alternating conflict-chain recoloring argument.

References

- [1] Gyárfás, A., Martin, R. R., Ruszinkó, M., & Sárközy, G. N. (2024). Proper edge colorings of planar graphs with rainbow C_4 -s. *Journal of Graph Theory*, 107(4), 833-846.
- [2] Kong, J., Wang, Y., & Zheng, M. (2026). B-Coloring of Planar Graphs. *Journal of Graph Theory*.

- [3] Gyárfás, A., & Sárközy, G. N. (2023). Proper Edge Colorings of Cartesian Products with Rainbow C_4 -s. *Graphs and Combinatorics*, 39(5), 98.
- [4] Chen, D., Wang, W., & Xu, Y. (2026). B-coloring of MP_2 -graphs. *Applied Mathematics and Computation*, 531, 130186.
- [5] Wu, F., Broersma, H., Zhang, S., & Li, B. (2024). Properly colored and rainbow C_4 -s in edge-colored graphs. *Journal of Graph Theory*, 105(1), 110-135.
- [6] He, J., & Han, M. (2024). Edge DP-Coloring in K_4 -Minor Free Graphs and Planar Graphs. *Axioms*, 13(6), 375.
- [7] Yin, H., Han, M., & Xu, M. (2023). Strong Edge Coloring of K_4 (t)-Minor Free Graphs. *Axioms*, 12(6), 556.
- [8] Yu, G., & Yu, R. (2023). Strong edge-coloring of 2-degenerate graphs. *Discrete Applied Mathematics*, 336, 11-14.
- [9] van Bommel, M. F., & Wang, P. (2026). Strong edge coloring algorithm for K_4 -minor free graphs. *Discrete Mathematics, Algorithms and Applications*, 18(05), 2550079.
- [10] Ye, J., Zou, J., & Han, M. (2025). Weak degeneracy of the square of K_4 -minor free graphs. *Applied Mathematics and Computation*, 500, 129439.
- [11] Chen, L., Nan, C., & Zhou, X. (2026). On the packing edge-colorings of subcubic K_4 -minor free graphs. *Discrete Applied Mathematics*, 393, 38-49.
- [12] Jumnongnit, P., Nakprasit, K., Ruksasakchai, W., & Sittitrai, P. (2025). Edge DP-coloring of planar graphs without 4-cycles and specific cycles. *Discrete Mathematics*, 348(4), 114353.
- [13] Zhang, L., Broersma, H., Lu, Y., & Zhang, S. (2025). List Edge Colorings of Planar Graphs without Non-induced 7-cycles. *Acta Mathematica Sinica, English Series*, 41(3), 1037-1054.
- [14] Wang, W. F., Wang, Y. Q., & Yang, W. S. (2024). Acyclic Edge Coloring of 1-planar Graphs without 4-cycles. *Acta Mathematicae Applicatae Sinica, English Series*, 40(1), 35-44.
- [15] Basavaraju, M., Hegde, S. M., & Kulamarva, S. (2023). Acyclic chromatic index of chordless graphs. *Discrete Mathematics*, 346(8), 113434.
- [16] Petruševski, M., & Škrekovski, R. (2024). Proper edge-colorings with a rich neighbor requirement. *Discrete Mathematics*, 347(3), 113803.
- [17] Cranston, D. W. (2023). Kempe equivalent list edge-colorings of planar graphs. *Discrete Mathematics*, 346(11), 113180.
- [18] Miao, Z., Song, Y., Wang, T., & Yu, X. (2023). List star edge coloring of generalized Halin graphs. *Discrete Mathematics*, 346(1), 113204.
- [19] Li, X., Li, Y., Lv, J. B., & Wang, T. (2023). Strong edge-colorings of sparse graphs with $3\Delta - 1$ colors. *Information Processing Letters*, 179, 106313.
- [20] Fu, J., & Lv, J. B. (2025). On injective edge-coloring of graphs with maximum degree 4. *Discrete Applied Mathematics*, 360, 119-130.
- [21] Lu, J., Liu, H., & Hu, X. (2024). On strong edge-coloring of graphs with maximum degree 5. *Discrete Applied Mathematics*, 344, 120-128.
- [22] Bradshaw, P., Clow, A., & Xu, J. (2025). Injective edge colorings of degenerate graphs and the oriented chromatic number. *European Journal of Combinatorics*, 127, 104139.
- [23] Cui, Q., & Han, Z. (2024). Injective edge-coloring of claw-free subcubic graphs. *Journal of Combinatorial Optimization*, 47(5), 87.
- [24] Tang, Y., Yin, H., & Han, M. (2023). Star edge coloring of K_2 , t-free planar graphs. *AIMS Mathematics*, 8(6), 13154-13161.
- [25] Wang, Y., Wang, Y., Wang, W., & Cui, S. (2022). Strong chromatic index of outerplanar graphs. *Axioms*, 11(4), 168.
- [26] Mattiolo, D., Mazzuoccolo, G., & Tabarelli, G. (2022). Graphs with large palette index. *Discrete Mathematics*, 345(5), 112814.

-
- [27] Dara, S., Mishra, S., Narayanan, N., & Tuza, Z. (2022). Strong edge coloring of Cayley graphs and some product graphs. *Graphs and Combinatorics*, 38(2), 51.
 - [28] Ma, B., & Yang, C. (2023). Distinguishing colorings of graphs and their subgraphs. *AIMS Mathematics*, 8(11), 26561-26573.
 - [29] Huang, N., & Chen, L. (2023). AVD edge-colorings of cubic Halin graphs. *AIMS Mathematics*, 8(11), 27820-27839.
 - [30] Casselgren, C. J., Małafiejski, M., Pastuszak, K., & Petrosyan, P. A. (2025). Near-interval edge colorings of graphs. *Discrete Applied Mathematics*, 372, 23-36.