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Article

Solving Partial Differential Equations for Wave Equation Problem on Strings by Applying the Fourier Transform

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Abstract. The wave equation on a string is an example of a partial differential equation problem. There are several methods for finding the solution to the wave equation on a string. The solution method will differ depending on the definition of the function's domain. This study aims to determine the form of solving the wave equation on the strings and the results of the analysis of the wave motion that depends on the number of boundary conditions, using a particular solution method, namely the Fourier transform method. The boundary conditions used are the Dirichlet boundary conditions. The Fourier transform method is used to obtain the solution of the wave equation on the string. The Fourier transform will transform the wave equation on the string and get the solution form of the wave equation on the string by applying the inverse Fourier transform. The results of this study obtained the same form of solution for each state from the wave equation on strings, namely in the form of the D'Alembert solution for the wave equation. As well, the movement of the wave will form a periodic solution by period 2π, with a different form of deviation occurring at each point x for each value t.

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1. Introduction

Differential equations are pivotal in the mathematical sciences, often employed to solve complex physical problems. By transforming physical phenomena and everyday occurrences into a mathematical framework, differential equations offer a systematic approach to understanding the world around us [1]. These equations are categorized into two main types: ordinary differential equations, which address simpler physical problems, and partial differential equations, capable of modeling more intricate and dynamic scenarios [2-5]. Among the myriad of physical phenomena that can be modeled through partial differential equations, the wave equation stands out, particularly in the context of string vibrations.

The wave equation, a quintessential example of a partial differential equation, plays a crucial role in understanding wave dynamics on strings. An essential aspect of tackling the wave equation is considering the initial and boundary conditions, as these elements significantly influence the choice of the solution method [3-5]. Various techniques have been developed for solving wave equations on strings, each suited to specific boundary conditions. The Laplace Transform method, for instance, is ideal for semi-infinite domains [6-8], while the method of separation of variables is tailored for finite domains, and D'Alembert's formulation is applicable to infinite domains [9-10].

In recent years, the Fourier Transform has emerged as a particularly versatile and effective method for solving partial differential equations, especially those with infinite domains. Moreover, the Fourier sine and cosine Transform methods extend this versatility to functions defined in finite and semi-infinite domains, respectively [11-13]. These methods are not only mathematically elegant but also lend themselves to computational efficiency, making them highly relevant in the era of high-performance computing.

The current study aims to explore the application of the Fourier Transform in solving the wave equation on strings. We focus on how different boundary conditions influence the wave dynamics and the efficacy of the Fourier Transform in capturing these nuances. This approach is not only significant in theoretical physics but also holds immense potential in practical applications, ranging from acoustics to materials science, where understanding wave behavior is fundamental. Recent advancements in computational techniques and the increasing availability of data have further accentuated the relevance of this method in contemporary research [14-19].

2. Literatur Review

The literature review section aims to provide a comprehensive overview of the foundational concepts and recent developments relevant to the study. This includes an in-depth analysis of the wave equation on strings, the crucial role of initial and boundary conditions, and the application of Fourier Transform methods in solving these equations.

2.1 Wave Equation on a String

The wave equation on strings is a classic example of a partial differential equation used to describe wave motion. It is essential in fields such as acoustics, material science, and physics. This equation models how disturbances on a string propagate over time, giving insights into the fundamental nature of wave dynamics. Recent studies have expanded on the traditional understanding of the wave equation, exploring nonlinear effects and complex boundary conditions that more accurately reflect real-world scenarios [20-24].

The wave equation on a string is the object study, whereas a piece of elastic string whose length with L on both ends is tied to the axis x in $x = 0$ and $x = L$. Then the string is pulled and red at a certain speed. If the strings are partitioned along the way Δx, it will look like the following image.

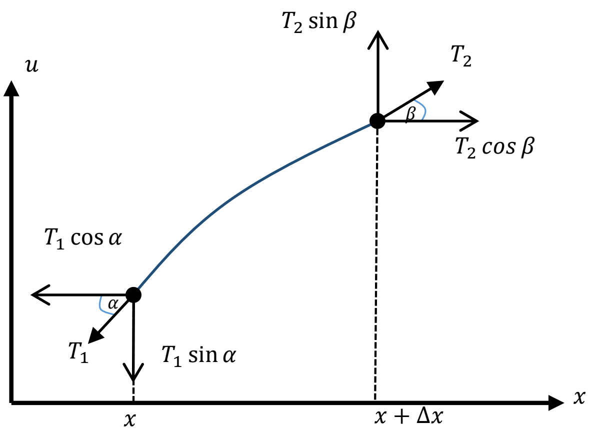

Figure 1. Partitions on a string

Referring to figure 1, then the general form of the wave equation on the string is obtained,

$$\frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{t}^2} = \boldsymbol{c}^2 \frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{x}^2}, \quad (1)$$

with, and $\boldsymbol{c} = \sqrt{\frac{\boldsymbol{T}}{\boldsymbol{\rho}}}$. Where $\boldsymbol{u}$ expresses the deviation of a wave, $\boldsymbol{x}$ is the point of occurrence of the variation, $\boldsymbol{t}$ represents time, $\boldsymbol{c}$ is a physical constant wwithnsions of velocity, $\boldsymbol{T}$ describes voltage, and $\boldsymbol{\rho}$ expresses the mass of strings.

2.2 Initial Conditions

Initial conditions in the context of the wave equation describe the state of the system at a specific initial time, usually denoted as $t = 0$. These conditions are essential for determining the starting point of the wave motion on a string. Typically, initial conditions are given in terms of the initial displacement and initial velocity of points on the string. Mathematically, these are expressed as:

2.2.1 Initial Displacement Condition: It specifies the shape of the string at $t = 0$, often denoted as $u(x, 0) = f(x)$, where $u(x, t)$is the displacement of the string at position x and time t, and f(x) is a predefined function.

2.2.2 Initial Velocity Condition: This defines the velocity of each point on the string at the initial time, expressed as $\frac{\partial t}{\partial u}(x, 0) = g(x)$, where $g(x)$ is a function specifying the initial velocity distribution along the string.

The initial conditions determine the physical state at the time t_0. The wave equation on the string is a second-order partial differential equation, so there will be two boundary conditions, namely:

$$u(x, 0) = f_1(x) \text{ dan } \frac{\partial}{\partial t}u(x, 0) = f_2(x) \quad (2)$$

2.3 Boundary Condition

Boundary conditions, on the other hand, specify the behavior of the string at its endpoints, which are crucial in determining how the wave is confined or reflected. There are generally three types of boundary conditions:

2.3.1 Fixed (Dirichlet) Boundary Conditions: These occur when the ends of the string are held fixed, implying that the displacement at these points remains zero throughout the motion. Mathematically, it is represented as $u(0,t)=u(L,t)=0$, where L is the length of the string.

2.3.2 Free (Neumann) Boundary Conditions: In this scenario, the endpoints of the string are free to move vertically, implying that the derivative of the displacement with respect to position is zero at the endpoints. This is expressed as $\frac{\partial x}{\partial u}(0,t) = \frac{\partial x}{\partial u}\ (L,t) = 0$.

2.3.3 Periodic Boundary Conditions: These conditions are used when the string forms a closed loop, implying that the behavior of the string is the same at both endpoints. This is represented by $u(0,t) == u(L,t)$ and often $\frac{\partial x}{\partial u}(0,t) = \frac{\partial x}{\partial u}\ (L,t) = 0$.

In this study, the Dirichlet boundary condition was used where the value of a function remains at its boundary [6]. Three circumstances are based on the sum of the boundary conditions, namely:

1. For strings with a finite length, there are two boundary conditions
$$u(0,t) = 0 \text{ dan } u(L,0) = 0, \tag{3}$$
2. For strings with semi-infinite length, there is one boundary condition
$$u(0,t) = 0\,, \tag{4}$$
3. Strings of infinite length have no boundary conditions

2.4 Fourier Transform Methods

The Fourier Transform is a powerful tool for analyzing and solving differential equations, particularly in the context of wave phenomena. Its ability to decompose complex waveforms into simpler sinusoidal components makes it highly effective for analyzing the wave equation on strings. Recent advancements in this area have led to more efficient algorithms and novel applications in computational physics and engineering, further underscoring the relevance of the Fourier Transform in modern scientific research [25-28].

3. Method To Finding Solution

This study employs a comprehensive literature review as its primary methodological approach. The focus is on gathering and scrutinizing a wide array of scholarly materials related to the Fourier Transform and its specific application, the Fourier Sinus Transform, in solving partial differential equations for wave equations on strings. This involves a meticulous examination of academic journals, books, conference proceedings, and recent publications in the field of mathematical physics, computational mathematics, and engineering. The method used is a literature study by finding and reviewing material regarding the Fourier Transform and the Fourier Sinus Transform methods.

3.1 Fourier's Transform

The Fourier Transform is a powerful mathematical tool that transforms a function of time (or space) into a function of frequency. In the context of wave equations, the Fourier Transform is used to decompose complex waveforms into their constituent frequencies. This decomposition facilitates the analysis and solution of differential equations by converting them from the time (or space) domain into the frequency domain, where they are often simpler to solve.

Recent research in this area has yielded significant advancements in the application of the Fourier Transform to partial differential equations. Innovative algorithms and computational methods have been developed to enhance the efficiency and accuracy of this approach, especially in dealing with complex boundary conditions and non-linear wave phenomena [29-31]. The Fourier Transform is applied to solve partial differential equations with intervals $(-\infty, \infty)$. The Fourier Transform of $\boldsymbol{u(x,t)}$ that $\boldsymbol{\bar{u}(\omega,t)}$ is defined as follows [32].

$$\boldsymbol{\bar{u}(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x} dx}, \tag{5}$$

and the inverse of the Fourier Transform is

$$\boldsymbol{u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{u}(\omega,t)\, e^{i\omega x} d\omega.} \tag{6}$$

3.2 Fourier Sinus Transform

The Fourier Sinus Transform, a variant of the Fourier Transform, is particularly suited for problems with specific boundary conditions, such as those encountered in wave equations on strings. This method is applied when the function to be transformed is odd, and thus, the transform predominantly uses sine functions. It is especially effective in handling problems with fixed or Dirichlet boundary conditions, where the displacement of the string is zero at the boundaries.

The latest research in the Fourier Sinus Transform has explored its broader applications, including its role in numerical methods for solving partial differential equations. Studies have also investigated its efficiency in computational simulations, offering new insights into wave dynamics and enhancing the ability to model real-world physical systems more accurately [33-36]. Fourier sine Transforms are commonly used for physical problems with semi-infinite domains or at intervals $(\mathbf{0},\infty)$.

Fourier sine Transform of $u(x,t)$ that $\bar{u}_s(\omega,t)$ is defined as follows [37].

$$\bar{u}_s(\omega,t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} u(x,t)\sin \omega x\, dx\,, \tag{7}$$

and the inverse of the Fourier sinus Transform is

$$\boldsymbol{u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \bar{u}_s(\omega,t)\sin \omega x\, d\omega\,,} \tag{8}$$

When the domain of finite physical problems or at intervals $(0,L)$ used finite Sine Fourier Transforms. Finite Sine Fourier Transforms of $u(x,t)$ that $\bar{u}_s(n,t)$ is the one defined as follows [37].

$$\bar{u}_s(n,t) = \int_0^{L} u(x,t)\sin\left(\frac{n\pi x}{L}\right) dx\,, \tag{9}$$

and the inverse of the finite Sine Fourier Transforms is

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty} \bar{u}_s(n,t)\sin\left(\frac{n\pi x}{L}\right), \tag{10}$$

4. Results and Discussion

This section presents the findings of the study, focusing on the application of the Fourier Transform method to solve the wave equation on a string under varying boundary conditions. It also includes visualizations to interpret the wave movements, providing a comprehensive understanding of the dynamics involved. The results of this study will show the solving of the wave equation on the string with three states of the number of boundary conditions solved using the Fourier Transform method, as well as visualization of the movement of the waves.

4.1 Solving the Wave Equation on a String with Finite Length

The application of the Fourier Transform to a string of finite length reveals distinct wave patterns [38-39]. This part of the study demonstrates how fixed boundary conditions at both ends of the string influence the wave behavior, resulting in standing wave patterns. The results highlight the formation of nodes and antinodes and their relation to the string's length and tension. Strings of finite length are strings that have a distance from $x = 0$ up to $x = L$. The deviations at both ends of the string are zero, so there are two boundary conditions as in equation (3).

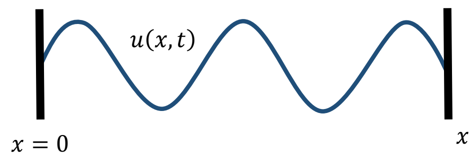

Figure 2. Strings of finite length

This problem can be solved using the finite Sine Fourier Transforms (9) and (10). Transform both segments of the wave equation (1), obtained

$$F_s\left(\frac{\partial^2}{\partial t^2}u(x,t)\right) = c^2 F_s\left(\frac{\partial^2}{\partial x^2}u(x,t)\right),$$

$$\frac{\partial^2}{\partial t^2}\bar{u}_s(n,t) = -\frac{c^2n^2\pi^2}{L^2}\,\bar{u}_s(n,t),$$

$$\frac{\partial^2}{\partial t^2}\bar{u}_s(n,t) + \frac{c^2n^2\pi^2}{L^2}\,\bar{u}_s(n,t) = 0, \tag{11}$$

thus obtained an ODE second-order (11), then a solution to the equation (11) is sought

$$\bar{u}_s(n,t) = c_1(n)\cos\left(\frac{cn\pi}{L}t\right) + c_2(n)\sin\left(\frac{cn\pi}{L}t\right). \tag{12}$$

Transform the initial conditions $f_1(x)$ and $f_2(x)$ obtained

$$\bar{u}_s(n,0) = F_{1s}(n) = \int_0^L f_1(x)\sin\frac{n\pi x}{L}\,dx,$$

$$\frac{\partial}{\partial t}\bar{u}_s(n,0) = F_{2s}(n) = \int_0^L f_2(x)\sin\frac{n\pi x}{L}\,dx,$$

then by using the initial condition transformation obtained

$$c_1(n) = F_{1s}(n) = \int_0^L f_1(x)\sin\frac{n\pi x}{L}\,dx,$$

$$c_2(n) = \frac{L}{cn\pi}F_{2s}(n) = \frac{L}{cn\pi}\int_0^L f_2(x)\sin\frac{n\pi x}{L}\,dx,$$

do the substitution $c_1(n)$ and $c_2(n)$ to the equation (12), gets

$$\bar{u}_s(n,t) = F_{1s}(n)\cos\left(\frac{cn\pi}{L}t\right) + \frac{L}{cn\pi}\,F_{2s}(n)\sin\left(\frac{cn\pi}{L}t\right), \tag{13}$$

Apply inverse of the finite Sine Fourier Transforms to (13) to obtain a particular solution

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left[F_{1s}(n)\cos\left(\frac{cn\pi}{L}t\right) + \frac{L}{cn\pi}\,F_{2s}(n)\,\sin\left(\frac{cn\pi}{L}t\right)\right]\sin\frac{n\pi x}{L}. \tag{14}$$

Now since, $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$ gets

$$\frac{2}{L}\sum_{n=1}^{\infty} F_{1s}(n)\cos\left(\frac{cn\pi}{L}t\right)\sin\left(\frac{n\pi x}{L}\right),$$

$$= \frac{1}{2}\left[\frac{2}{L}\sum_{n=1}^{\infty} F_{1s}(n)\sin\frac{n\pi}{L}(x+ct) + \frac{2}{L}\sum_{n=1}^{\infty} F_{1s}(n)\sin\frac{n\pi}{L}(x-ct)\right],$$

$$= \frac{1}{2}\,[f_1(x+ct) + f_1(x-ct)].$$

And by applying $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$ and $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, gets

$$\frac{2}{L}\sum_{n=1}^{\infty}\frac{L}{cn\pi}\,F_{2s}(n)\sin\left(\frac{cn\pi}{L}t\right)\sin\left(\frac{n\pi x}{L}\right)$$
$$=\frac{2}{L}\sum_{n=1}^{\infty}\frac{L}{cn\pi}\,F_{2s}(n)\left(\frac{e^{\frac{in\pi}{L}ct}-e^{-\frac{in\pi}{L}ct}}{2i}\right)\left(\frac{e^{\frac{in\pi}{L}x}-e^{-\frac{in\pi}{L}x}}{2i}\right)$$
$$=\frac{1}{c}\,\frac{2}{L}\sum_{n=1}^{\infty}\frac{L}{n\pi}\,F_{2s}(n)\left[-\frac{1}{2}\left(\frac{e^{\frac{in\pi}{L}(x+ct)}+e^{-\frac{in\pi}{L}(x+ct)}}{2}\right)+\frac{1}{2}\left(\frac{e^{\frac{in\pi}{L}(x-ct)}+e^{-\frac{in\pi}{L}(x-ct)}}{2}\right)\right]$$
$$=\frac{1}{2c}\,\frac{2}{L}\sum_{n=1}^{\infty}F_{2s}(n)\left[\frac{-\cos\frac{n\pi}{L}(x+ct)+\cos\frac{n\pi}{L}(x-ct)}{\frac{n\pi}{L}}\right]$$
$$=\frac{1}{2c}\,\frac{2}{L}\sum_{n=1}^{\infty}F_{2s}(n)\left[\int_{x-ct}^{x+ct}\sin\left(\frac{n\pi}{L}r\right)dr\right],$$
$$=\frac{1}{2c}\int_{x-ct}^{x+ct}\frac{2}{L}\sum_{n=1}^{\infty}F_{2s}(n)\sin\left(\frac{n\pi}{L}r\right)dr,$$
$$=\frac{1}{2c}\int_{x-ct}^{x+ct}f_2(r)\,dr.$$

Then by applying the trigonometric formula and the Euler formula on (14), a settlement is obtained in the form of the D'Alembert formula,

$$u(x,t)=\frac{1}{2}\,[f_1(x+ct)+f_1(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}f_2(r)\,dr. \tag{15}$$

4.2 Solving the Wave Equation on a String with Semi-Infinite Length

In this subsection, the analysis extends to a string with one fixed end and another extending infinitely. The Fourier Transform method reveals how waves propagate in such a medium, with a focus on the damping and reflection phenomena at the fixed boundary. This case study is particularly relevant for understanding wave propagation in semi-bounded media [40-41]. Strings with semi-infinite lengths are strings with a distance from $x = 0$ up to $x = \infty$. The deviation at $x = 0$ zero, so there is one boundary condition as in equation (4).

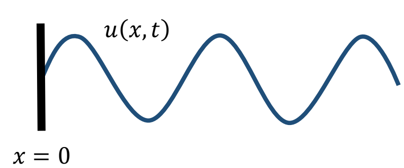

Figure 3. Strings of semi-infinite length

This problem can be solved using the Sine Fourier Transforms (7) and (8). The completion step is similar to the completion step for a finite string length, and the difference lies only in the form of the method used. Thus obtain a particular form of settlement.

$$u(x,t)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\left[F_{1s}(\omega)\cos\omega ct+\frac{F_{2s}(\omega)}{\omega c}\sin\omega ct\right]\sin\omega x\,d\omega, \tag{16}$$

applying trigonometric formulas and Euler's formulas obtained a solution in the form of d'Alembert's procedure similar to equation (15).

4.3 Solving the Wave Equation on a String with Infinite Length

Here, the infinite length of the string offers a unique scenario where the waves are not bound by any physical constraints. The study explores how wave packets propagate and disperse over time and space, using the Fourier Transform to analyze the wave behavior in an unbounded domain [42-44]. Strings of infinite length are strings with a distance of $x = -\infty$ up to $x = \infty$. Strings with this condition have no boundary conditions.

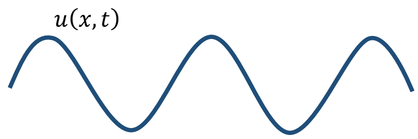

Figure 4. Strings of infinite length

This problem can be solved using Fourier Transforms (5) and (6). The completion step is similar to the completion step for a finite string length, and the difference lies only in the form of the method used. Therefore, get a particular form of completion.

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[F_1(\omega) \cos \omega ct + \frac{F_2(\omega)}{\omega c} \sin \omega ct \right] e^{i\omega x} d\omega, \tag{17}$$

applying Euler's formula obtained a solution in the form of a D'Alembert formula similar to equation (15).

4.4 Visualization and Interpretation of Wave Movements with Finite String Lengths

Visualizations of wave movements on a finite string are provided, illustrating the formation of standing waves, nodes, and antinodes [45]. These visualizations aid in understanding the harmonic patterns formed and how they are influenced by the string's physical properties. Example of a problem on the strings whose two ends are tied in $0 < x < 2$. The strings are vibrated by providing the initial deviation and the initial transverse velocity.

$$f_1(x) = 2x - 3 \text{ dan } f_2(x) = 0.$$

The particular solution of such wave equations is equation (14) by obtaining, $L = 2$

$$u(x,t) = \frac{2}{2} \sum_{n=1}^{\infty} \left[F_{1s}(n) \cos\left(\frac{cn\pi}{2} t\right) + \frac{2}{cn\pi} F_{2s}(n) \sin\left(\frac{cn\pi}{2} t\right) \right] \sin \frac{n\pi x}{2},$$

by substituting the initial conditions $f_1(x)$ and $f_2(x)$ obtaining

$$F_{1s}(n) = \int_0^2 (2x-3) \sin \frac{n\pi x}{2} dx = -\frac{2}{n\pi} \cos n\pi - \frac{6}{n\pi} + \frac{8}{n^2\pi^2} \sin n\pi\,,$$

$$F_{2s}(n) = \int_0^2 f_2(x) \sin \frac{n\pi x}{2} dx = 0.$$

Thus obtained

$$u(x,t) = \sum_{n=1}^{\infty} \left(-\frac{2}{n\pi} \cos n\pi - \frac{6}{n\pi} + \frac{8}{n^2\pi^2} \sin n\pi \right) \cos\left(\frac{cn\pi}{2} t\right) \sin \frac{n\pi x}{2}.$$

Shown Visualization of the movement of waves by taking the value of $c = 1$. Figure 5 shows the sign of the wave when $0 < t < 2$, and figure 6, shows the movement of the surge in the two-dimensional form at various specific times.

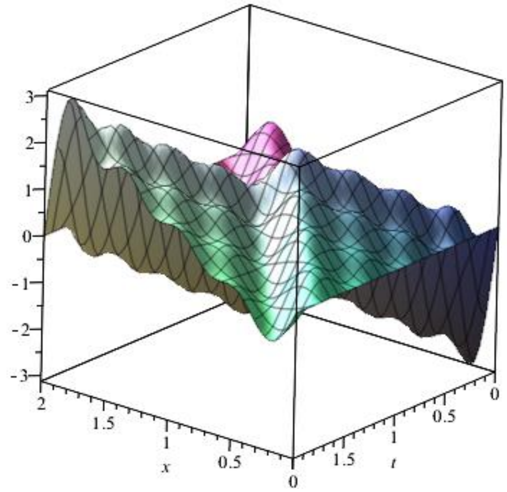

Figure 5. Deviation visualization $\boldsymbol{f_1(x) = 2x - 3}$

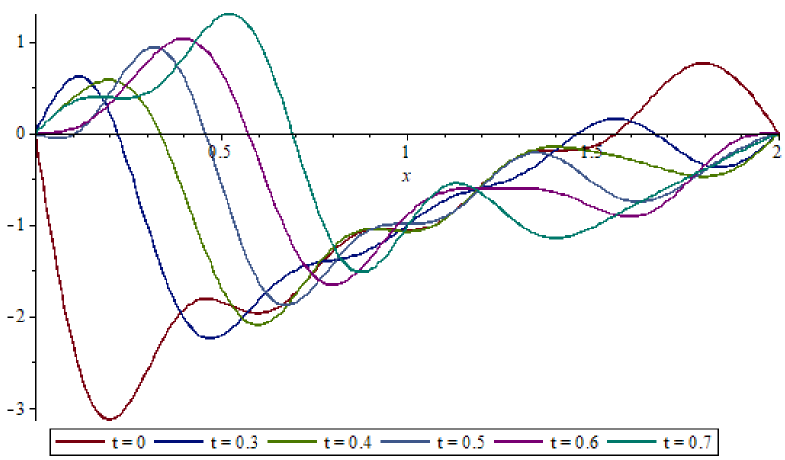

Figure 6. Wave movement at various times $\boldsymbol{f_1(x) = 2x - 3}$

At each t given, one seen the difference in wave deviation that occurs at each point x. Then $t = 0$, the wave deviation form is the initial deviation $f_1(x) = 2x - 3$ indicated by a red line. For others t, the one user has a wave deviation shape that is different from the initial deviation. There is no deviation at the point $x = 0$ and $x = 2$, and this happens because the point $x = 0$ and $x = 2$ is the end point of the bound string, so the divergence is $u(0, t) = 0$ and $u(2, t) = 0$.

4.5 Visualization and Interpretation of Wave Movements with Semi-Infinite String Lengths

This subsection includes graphical representations of wave propagation along a semi-infinite string, highlighting the interaction of incident and reflected waves at the fixed boundary. The visualizations help in comprehending the complex dynamics of wave reflection and transmission [46-47]. Given an example of a problem on a string in which one end is bound at an interval $0 < x < \infty$. The strings are vibrated by providing the initial deviation and the initial transverse velocity.

$$f_1(x) = \sin x \quad \text{dan} \quad f_2(x) = 0.$$

The solution of the wave equation uses the following formula of D'Alembert (15),

$$u(x,t) = \frac{1}{2}\,[f_1(x + ct) + f_1(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} f_2(r)\,dr,$$

by substituting the initial conditions $f_1(x)$ and $f_2(x)$ obtaining

$$u(x,t) = \frac{1}{2}\,[\sin(x+ct) + \sin(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} 0\,dr.$$

Thus obtained

$$u(x,t) = \sin(x)\cos(ct).$$

Visualization of the movement of waves by taking values $c = 5$ at intervals $0 < x < 7$. Figure 7 the movement of the wave when $0 < t < 2$, and figure 8 shows the sign of the wave in the two-dimensional form at various specific times.

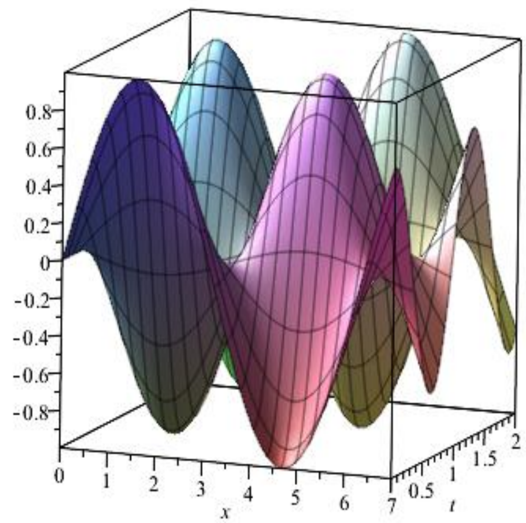

Figure 7. Deviation visualization $\boldsymbol{f_1(x) = sin\,x}$

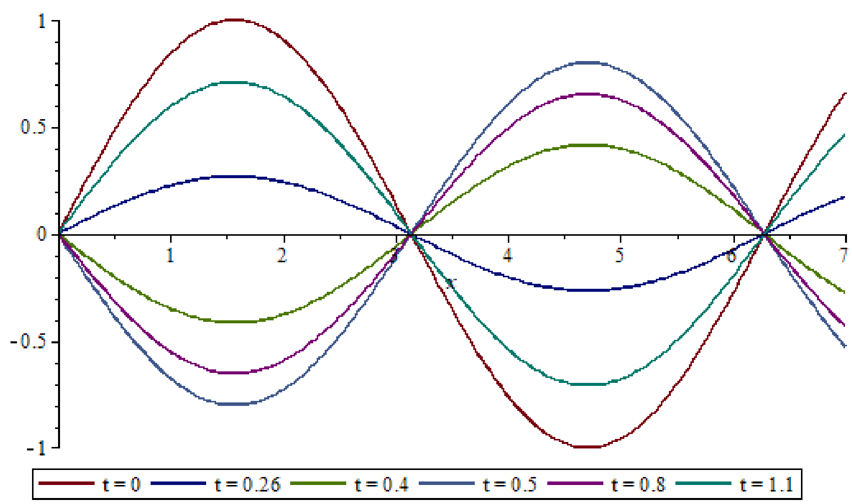

Figure 8. Wave movement at various times $\boldsymbol{f_1(x) = sin\,x}$

An intersection between the deviations and axes x occurs at the point $x = 3{,}14$ and $x = 6{,}28$, or it can also be written as $x = \pi$ and $x = 2\pi$. No deviation accurs at the point $x = 0$ because it is the end point of the string bound, so that $u(0,t) = 0$. Since the initial condition chosen is $f_1(x) = \sin x$, which will always intersect the axis x at the point $x = n\pi$ with $n = 0, \pm 1, \pm 2, \ldots$, then these strings will intersect with the axis x at the point $x = 0$, $x = \pi$, and it is multiple.

4.6 Visualization and Interpretation of Wave Movements with Infinite String Lengths

Finally, the infinite string case is visualized to depict wave packet propagation and dispersion. These illustrations provide insights into the behavior of unbounded wave propagation and the role of dispersion in wave dynamics [48-49]. Given an example of a problem on strings whose two ends are not tied at intervals $-\infty < x < \infty$. The strings are vibrated by providing the initial deviation and the initial transverse velocity.

$$f_1(x) = 0 \text{ dan } f_2(x) = \sin 2x + \cos\frac{x}{2}.$$

The solution of the wave equation uses the following formula of D'Alembert (15),

$$u(x,t) = \frac{1}{2}\,[f_1(x+ct) + f_1(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} f_2(r)\,dr,$$

by substituting the initial conditions $f_1(x)$ and $f_2(x)$ obtaining

$$(x,t) = \frac{1}{2}\,[0] + \frac{1}{2c}\int_{x-ct}^{x+ct}\left(\sin 2r + \cos\frac{r}{2}\right)dr.$$

Thus obtained

$$u(x,t) = \frac{1}{2c}\left[\sin(2x)\sin(2ct) + 4\cos\left(\frac{x}{2}\right)\sin\left(\frac{ct}{2}\right)\right].$$

Visualization of the movement of waves by taking values $c = 10$ at intervals $-5 < x < 5$. Figure 9 shows the sign of the wave when $0 < t < 3$, and for figure 10 the movement of the wave in the two-dimensional form at various specific times.

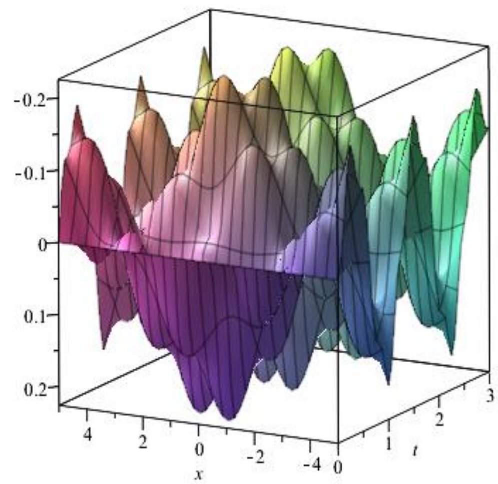

Figure 9. Deviation visualization $f_2(x) = \sin 2x + \cos\frac{x}{2}$

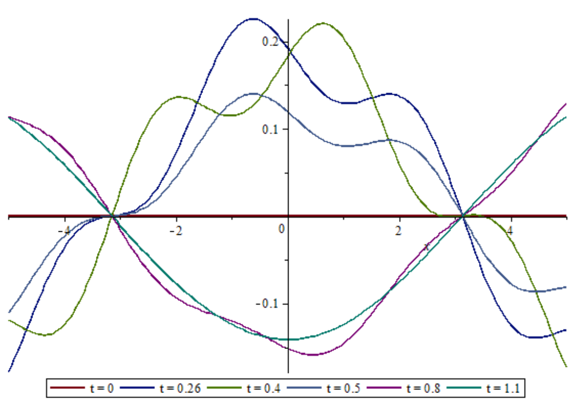

Figure 10. Wave movement at various times $f_2(x) = \sin 2x + \cos\frac{x}{2}$

There are differences in the form of deviations that occur at each point x for a given value t. There are two points of intersection, namely $x = -3{,}14$ and $x = 3{,}14$ or $x = -\pi$ and $x = \pi$. If the interval is extended, there will be another intersecting point with the axis x, at a point that is a multiple of $x = -\pi$ and $x = \pi$.

5. Conclusion

There are three states of the equation of the wave equation on the string based on giving the number of its boundary conditions. The resolution of the three states is obtained by applying three methods related to the Fourier Transform. The wave equation on a string with three forms of boundary conditions is solved using the finite dan infinite Sine Fourier Transform and Fourier Transform method. The three states of the wave equation on the string produce the same solution, i.e. in the form of D'Alembert's solution to the wave equation. As well as, there are differences in the form of deviations that occur at each point x for each value t, and the movement of waves will form a periodic solution with periods 2π.

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