Eksakta : Berkala Ilmiah Bidang MIPA
VOLUME 23 NO 03 2022, pp 223-230
ISSN : Print 1411-3724 — Online 2549-7464
DOI : https://doi.org/10.24036/eksakta/vol23-iss03/331

Eksakta
Berkala Ilmiah Bidang MIPA
http://www.eksakta.ppj.unp.ac.id/index.php/eksakta

Article

Grundwald-Letnikov Operator and Its Role in Solving Fractional Differential Equations

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Article Info

Article history :

Received August 08, 2022
Revised August 25, 2022
Accepted September 04, 2022
Published September 30, 2022

Keywords :

Fractional, Grundwal-Letnikov, Riemann-Liouville, Caputo, Homotopy.

Abstract. Leibnitz in 1663 introduced the derivative notation for the order of natural numbers, and then the idea of fractional derivatives appeared. Only a century later, this idea began to be realized with the discovery of the concepts of fractional derivatives by several mathematicians, including Riemann (1832), Grundwal, Fourier, and Caputo in 1969. The concepts in the definitions of fractional derivatives by Riemann-Liouville and Caputo are more frequently used than other definitions, this paper will discuss the Grunwald-Letnikov (GL) operator, which has been discovered in 1867. This concept is less popular when compared to the Riemann-Liouville and Caputo concepts, however, this concept is quite interesting because the concept of derivation is developed from the definition of ordinary derivatives. In this paper will be shown that the formulas for the fractional derivative using the GL concept are the same as the results obtained using the Riemann-Liouville and Caputo concepts. As a complement, we will give an example of solving a fractional differential equation using Modified Homotopy Perturbation Methods.

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1. Introduction

There are so many problems which the solutions use derivatives, for example, in the application of the first derivative, in terms of finding the slope of a tangent line, changing a vehicle speed, even the average growth rate of a population. In economics to determine the marginal cost of a total production cost function to produce a number of goods issued by a company. To determine the maximum and minimum values, the second derivative of a function is used. However, all of these problems only use the 1st, 2nd, 3rd derivatives, which means that so far what is known is only derivatives with the level or order of a natural number. In accordance with the development of science, nowadays there are many problems that require derivatives with the order of rational numbers, for example in modeling and control systems of dynamic programming, in energy problems, in railroad signals, determining option prices in economics, and others. The derivative with the order of rational numbers then called the fractional derivative.

In the middle of the 19th century, mathematicians Riemann (1826-1866) and Liouville (1809-1882) put forward the concept of α order fractional integral as a generalization of the derivative proposed by Leibniz (1646-1716). Furthermore, Grundwald and Letnikov (1867), Fourier, and others including the last Caputo (1969) seemed to be competing to create and develop a definition of fractional derivative. The Riemann-Liouville and Caputo versions are more popular than the others, this is because they are more widely used by researchers, especially in the applied field.

Currently, the development of research on fractions is so rapid, both grouped in Fractional Integrals and Derivatives [1,2], and in the Calculus Fractional group [3]. Fractional models are also very diverse, including: Diffusion-Reaction Equation model and Fractional Gas Dynamics Equation model [4,10]. The most widely used fractional operators are Riemann-Liouville and Caputo operators, only a few use Grundwald-Letnikov as in [5,6,13]. To solve fractional differential equations, many methods are used, including Homotopy Perturbation Method [7], Modified Homotopy Perturbation Method [8,9], and Homotopy Perturbation Method using Sumudu Transformation [10]. Local Fractional [11,12], applied Neural Network [13] and viscosity analysis [14] are the last things that are used as references for this paper.

This paper, the Grundwald-Letnikow version of the fractional derivative operator is presented, with the aim of showing similarities with other versions, both in terms of the existence of a well-defined definition, determining the basic derivative formula, even in its application to solve a problem in fractional differential equation. The method used to determine the solution is the Modified Homotopy Perturbation Method.

2. Literatur Review

In this section, several definitions related with the subject will be presented. These include Gamma functions, Floor and Ceiling functions, Mittag-Lefler functions, Laplace transforms, and fractional derivatives

Definition 2.1. Gamma Function, written as $\Gamma(n)$ defined as

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \quad ; n > 0 \; .$$

Theorem 2.2. If $\Gamma(n)$ is Gamma Function, so it call

1. $\Gamma(n+1) = n\Gamma(n) \; ; n > 0$ dengan $\Gamma(1) = 1$
2. $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}; \;\; n < 0$
3. $\Gamma(n) = (n-1)! \;\; ; n = 1, 2, 3, \ldots$

Definition 2.3. Floor and Ceiling Function

1. If $x \in \mathbf{R}$ is between two integers. Ceiling function of x denoted by $\lceil x \rceil$ represents the smallest integer greater than or equal to x.
2. If $x \in \mathbf{R}$ is between two integers. Floor function from x denoted by $\lfloor x \rfloor$ represents the largest integer that is less than or equal to x.

Definition 2.4. Riemann Liouville

Riemann-Liouville Operator Fractional Derivative is D_x^α , where

$$D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x f(t)\,(x-t)^{-(\alpha-n+1)}\,dt \qquad (1)$$

with $n-1 \le \alpha < n$ or $n-1 = \lfloor \alpha \rfloor$.

Definition 2.5. Caputo Operator

Let α is a real number, and $n-1 < \alpha \le n$ where n is natural number. Fractional derivative of $f(x)$ with oder α is

$$D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x f(t)\,(x-t)^{n-\alpha-1} f^{(n)}(t)dt\,. \qquad (2)$$

There is an example of how to determine the fractional derivative using the definition of the Riemann-Liouville operator. Example:

Fractional derivative of $f(x) = x^2$ with orde $\alpha = \frac{1}{2} \in \mathbf{R}$ is, so

$$_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x f(t)(x-t)^{-(\alpha-n+1)}dt \;\;;\;\; n-1 = \lfloor \alpha \rfloor = \left\lfloor \frac{1}{2} \right\rfloor = 0 \;\;\text{so}\;\; n=1$$

$$_0D_x^{\frac{1}{2}} x^2 = \frac{1}{\Gamma\left(1-\frac{1}{2}\right)}\left(\frac{d}{dx}\right)^1 \int_0^x t^2 (x-t)^{-\left(\frac{1}{2}-1+1\right)} dt$$

$$= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{d}{dx} \int_0^x t^2 (x-t)^{-\frac{1}{2}} dt$$

$$= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{d}{dx}\left(-2t^2(x-t)^{\frac{1}{2}} - \frac{8}{3}t(x-t)^{\frac{3}{2}} - \frac{16}{15}(x-t)^{\frac{5}{2}} \Big|_0^x \right)$$

$$= \frac{1}{\sqrt{\pi}} \frac{d}{dx}\left(2\cdot 0^2(x-0)^{\frac{1}{2}} + \frac{8}{3} 0\cdot(x-0)^{\frac{3}{2}} + \frac{16}{15}(x-0)^{\frac{5}{2}} \right)$$

$$_0D_x^{\frac{1}{2}} x^2 = \frac{1}{\sqrt{\pi}} \frac{d}{dx}\left(\frac{16}{15} x^{\frac{5}{2}} \right) = \frac{1}{\sqrt{\pi}} \frac{8}{3} x^{\frac{3}{2}}$$

If the example above is done using the Caputo operator, the results will be the same. The result will different if $f(x)$ in the form of constant function, where the result using Caputo Operator is zero, while the result with Riemann-Liouville is not zero. This is clearly seen from the formula in definition 2.5. $f^{(n)}(t) = 0$.

Theorem 2.6. Linearity of Operators

If f and g is a function and D_x^α is the fractional derivatives operator with order-α , then

$$D_x^\alpha\,[a\,f(x) + b\,g(x)] = a\,D_x^\alpha\,f(x) + b\,D_x^\alpha\,f(x)\,.$$

Definition 2.7. Fractional Differential Equation

One of the general forms of a fractional differential equation of order α is

$$\frac{d^\alpha u}{dt^\alpha} + a\,u(t) + b\,u^2(t) = A(t) \qquad (3)$$

with $0 < \alpha \le 1$, $t > 0$, a, b the real constant, and the initial condition is $u(0) = 0$.

3. Method

Many methods can be used to solve this fractional differential equation, including the Modified Homotopy Pertubation Method. In this section, The Modified Homotopy Pertubation Method to solve Fractional Differential Equation model will be elaborated.

The Fractional Differential Equation Model is given below:

$$D^\alpha u(t) + L(u(t)) + N\big(u(t)\big) = A(t), \;\; t > 0, \qquad m-1 < \alpha \le m \qquad (4)$$

where D^α fractional derivative with order α, L is linear operator, N is nonlinear operator, and $A(t)$ is a function of t. The initial condition is:

$$u^k(0) = c_k, \qquad k = 0, 1, 2, \dots, m-1 \,. \qquad (5)$$

From (2), we obtained the Homotopy equation

$$u^m + L(u) - A(t) = p\,[u^m - N(u) - D^\alpha u], \qquad p \in [0,1] \qquad (6)$$

or

$$u^m - A(t) = p\,[u^m - L(u) - N(u) - D^\alpha u], \qquad p \in [0,1]\,. \qquad (7)$$

Hence the solution of (6) and (7) in form of p-power series is

$$u = u_0 + p\,u_1 + p\,u_1 + p^2\,u_2 + p^3\,u_3 + \cdots. \qquad (8)$$

By substituting (8) to (7) and take $p = 1$, then solution function of (6) will be

$$u(t) = \sum_{n=0}^{\infty} u_n(t) \qquad (9)$$

where $u_n(t)$ is obtained from integral result of derivative function as follows

$$\left.\begin{aligned}
&\frac{d^m u_0}{dt^m} = A(t), \;\; u^k(0) = c_k \\
&\frac{d^m u_1}{dt^m} = \frac{d^m u_0}{dt^m} - L_0(u_0) - N_0(u_0) - D^\alpha u_0, \qquad u^k(0) = c_k \\
&\frac{d^m u_2}{dt^m} = \frac{d^m u_1}{dt^m} - L_1(u_0\,, u_1) - N_1(u_0\,, u_1) - D^\alpha u_{01}, \qquad u^k(0) = c_k \\
&\quad\vdots
\end{aligned}\right\} \qquad (10)$$

4. Results and Discussion

The results of the research that will be presented in this paper are the derivation of the Grundwald-Letnikov operator formula, the basic theorem on the general formula for fractional derivatives using the Grundwald-Letnikov operator, and examples of solving fractional differential equations using the Modified Homotopy Pertubation Method.

4.1. Grunwald-Letnikov

In contrast to Riemann-Liouville and Caputo who defined a fractional derivative through integration, Grunwald-Letnikov formulated a fractional derivative as a generalization of derivatives of the order of natural numbers.

Theorem 2 : Fractional derivative of $f(x)$ with order α in interval $[a\ ,\ b]$ is

$$D_x^{\alpha} f(x) = \lim_{h\to 0} \frac{1}{h^n} \sum_{i=0}^{n} (-1)^i \frac{\Gamma(\alpha+1)}{\Gamma(i+1)\,\Gamma(\alpha-i+1)} f(x-ih) \tag{11}$$

where $n = \left\lfloor \frac{b-a}{n} \right\rfloor$.

Proof:

First, we know that the first derivative of the function $y = f(x)$ is

$$f\,'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Hence second derivative is

$$f''(x) = \lim_{h\to\infty} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to\infty} \frac{f(x+2h) - 2\,f(x+h) + f(x)}{h^2}$$

$$= \lim_{h\to\infty} \frac{1}{h^2} \sum_{i=0}^{2} (-1)^i \binom{2}{i} f(x+ih).$$

So the derivative for n^{th}

$$f^{(n)}(x) = \lim_{h\to 0} \frac{1}{h^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x+ih), \text{ where } \binom{n}{i} = \frac{n!}{(n-i)!\,.i!}.$$

By generalizing the order of natural numbers n to the fractional order α and utilizing the definition of the Gamma function, it is obtained that the fractional derivative of a function f (x) with order-α is

$$D_x^{\alpha} f(x) = \lim_{h\to 0} \frac{1}{h^n} \sum_{i=0}^{n} (-1)^i \frac{\Gamma(\alpha+1)}{\Gamma(i+1)\,\Gamma(\alpha-i+1)} f(x-ih).$$

So That theorem was proved.

Furthermore, it is shown that the results of Grundwal-Letnikov are similar to those of Riemann-Liouville and Caputo on the basic formula of the derivative.

Theorem 3 : Derivative order- $\alpha \in \mathbf{N}$ from function $f(x) = x^p$ to x is

$$D_x^\alpha \ x^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} \ x^{p-\alpha} \ . \tag{12}$$

Proof: Using the definition of the Grunwald-Letnikov derivative, so

$$f^{(n)}(x) = \lim_{h\to 0} \frac{1}{h^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x+ih) = p(p-1)(p-2)\cdots(p-n+1)x^{p-n} \ .$$

Thus the n^{th} derivative of the function $f(x) = x^p$ to x is

$$\begin{aligned} f^{(n)}(x) &= \lim_{h\to 0} \frac{1}{h^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x+ih) \\ &= p(p-1)(p-2)...(p-n-1)x^{p-n} \\ &= \frac{p!}{(p-n)!} x^{p-n} \\ &= \frac{\Gamma(p+1)}{\Gamma(p-n+1)} x^{p-n}. \end{aligned}$$

The last thing that will be explained in this paper is solving a Fractional Differential Equation using the Modified Homotopy Perturbation Method.

For example (3). If we take $a = -1$, $b = 1$, and $A(t) = 2t$, the equation become

$$\frac{d^\alpha u}{dt^\alpha} - u(t) + u^2(t) = 2t \ , \ t > 0, \tag{13}$$

with initial conditions $u(0) = 0$.
From (4) we have $L\big(u(t)\big) = -u$ and $N\big(u(t)\big) = u^2$.

Based on (7), homotopy equation is

$$u' - 2t = p[u' + u - u^2 - D^\alpha u], \qquad p \in [0,1]$$

and based assumption for the solution to homotopy equation is

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots \ .$$

By substituting the basic assumptions and initial conditions to the homotopy equation, (10) bacame

$$\begin{aligned} &u_0' = 2t, \qquad u_0(0) = 0 \\ &u_1' = u_0' + u_0 - u_0^2 - D^{\boldsymbol{\alpha}} u_0, \qquad u_1(0) = 0 \\ &u_2' = u_1' + u_1 - 2u_0u_1 - D^{\boldsymbol{\alpha}} u_1, \qquad u_2(0) = 0 \\ &u_3' = u_2' + u_2 - 2u_0u_2 - u_1^2 - D^{\boldsymbol{\alpha}} u_2, \qquad u_3(0) = 0. \\ &\ \vdots \end{aligned}$$

Hence, we obtained

$$\begin{aligned} &u_0 = t^2, \\ &u_1 = t^2 + \frac{t^3}{3} - \frac{t^5}{5} - \frac{2t^{3-\alpha}}{\Gamma(4-\alpha)} \end{aligned}$$

$$u_2 = t^2 + \frac{2t^3}{3} + \frac{t^4}{12} - \frac{3t^5}{5} - \frac{13t^6}{90} + \frac{t^8}{20} - \frac{4t^{3-\alpha}}{\Gamma(4-\alpha)} - \frac{2t^{4-\alpha}}{\Gamma(5-\alpha)} + \frac{24t^{6-\alpha}}{\Gamma(7-\alpha)} + \frac{2t^{4-2\alpha}}{\Gamma(5-2\alpha)} + \frac{4t^{7-2\alpha}}{(7-2\alpha)\Gamma(4-\alpha)}$$

etc.

Therefore, solution function is

$$\begin{aligned} u(t) &= u_0(t) + u_1(t) + u_2(t) + \cdots \\ &= 3t^2 + t^3 + \frac{t^4}{12} - \frac{4t^5}{5} - \frac{13t^6}{90} + \frac{t^8}{20} - \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} - \frac{2t^{4-\alpha}}{\Gamma(5-\alpha)} + \frac{24t^{6-\alpha}}{\Gamma(7-\alpha)} \\ &\quad + \frac{2t^{4-2\alpha}}{\Gamma(5-2\alpha)} + \frac{4t^{7-2\alpha}}{(7-2\alpha)\Gamma(4-\alpha)} + \cdots \end{aligned}$$

with a graph of the solution function as below.

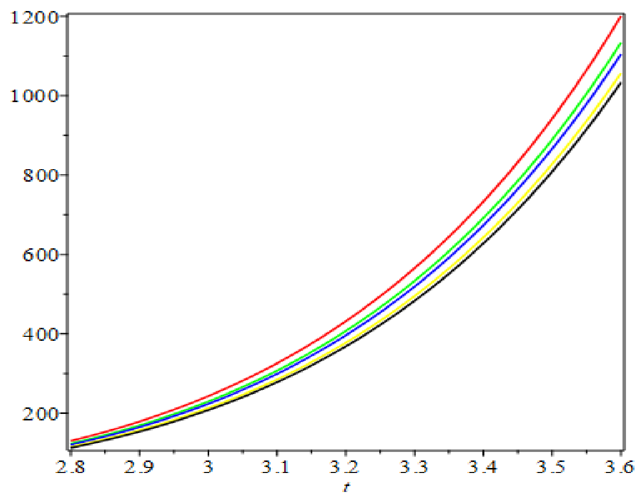

Figure 1. Graphs of Solution Function F

5. Conclusion

Based on what has been described in this paper, our first conclusion is that the Grundwald-Letnikov fractional derivative operator is well defined which has been demonstrated by induction through derivatives of the natural number order. The second conclusion is that the derivation of the basic derivative formula obtained by using the Grundwald-Letnikov operator is the same as that obtained by Riemann-Liouville and Caputo. As the final conclusion of this paper, we get that the three operators discussed can be used to solve fractional differential equations, especially through the Modified Homotopy Parturbation Method.

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